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IMPULSIVE MOTION OF A SUSPENSION: EFFECT OF ANTISYMMETRIC STRESSES AND PARTICLE ROTATION

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Abstract—A two-phase continuum theory (two-fluid model) for a suspension of rigid spherical particles in a Newtonian fluid is applied to investigate theoretically the flow induced by impulsive motion of an infinite flat plate. Consideration of rotational intertia of the particles gives rise to an antisymmetric part of the volume averaged stress tensor of the continuous phase. The influence of particle rotation and of antisymmetric stresses of the continuous phase, which depend on the relative rotational motion between the particles and the ambient fluid, on the motion of each phase and on the skin friction is examined.

Approximate solutions to the equations, corresponding to the physical situation of large and small particle slip, are obtained by power series expansions for small and large times.

1. INTRODUCTION

The problem of the flow induced by the impulsive motion of an infinite flat plate parallel to its own plane, first considered by Stokes for a Newtonian fluid, has been extended to two-phase flows by several authors. Liu (1967), Marble (1970) and Healy & Yang (1972) studied the flow induced in a dusty gas (rigid spherical particles in a gas). The interphase force, i.e. the force on the dispersed phase due to the presence and motion of the continuous phase includes in their analysis the Stokes drag force, the volume fraction of the dispersed phase being very small. Murray (1967), in addition to the drag force, introduces the added mass effect in the interphase force. Otterman (1968) takes into account the lateral migration of the particles in the shear flow of the continuous phase by introducing the slip-shear force of Saffman (1965) besides the drag force. Di Giovanni & Lee (1974) have extended Otterman's results by including the added mass effect and finite volume fraction of the dispersed phase. None of the above cited papers takes into account antisymmetric stresses of the continuous phase, which arise as a result of consideration of rotational inertia of the rigid particles (e.g. Afanas'ev & Nikolaevskii 1969 and Immich 1980a, b). Hamed & Tabakoff (1973, 1974, 1975) were the first to include in their analysis of the impulsive motion of an infinite flat plate antisymmetric stresses due to the relative rotational motion between the particles and the fluid. The interphase force includes the drag force and the slip-shear lift force of Saffman (1965).

However, this lift force on the particles in their equations is introduced in the wrong direction. Hence, their result of a demixed region near the plate due to particle migration away from the wall is in direct contradiction to the results of particle migration to the wall for the present problem, as obtained by Otterman (1968), Otterman & Lee (1970), Di Giovanni & Lee (1974) and Immich (1979). Furthermore, the symmetric part of the stress tensor of the continuous phase in the equations of Hamed and Tabakoff is contained in the same form as for a pure fluid, though, e.g. Drew & Segel (1971) and Ishii (1975) have shown that the stress tensor in the momentum balance of the continuous phase appears in a form multiplied with the volume fraction of the continuous phase.

In the present paper, the impulsive motion of an infinite flat plate is examined by application of a two-phase continuum theory as derived by the author (Immich 1979, 1980a, b) by means of a volume-averaging method. Particle rotation and antisymmetric stresses are considered, the interphase force includes the pressure force, the drag force, the added mass effect and the slip-shear lift force of Saffman (1965).

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The purpose of the present paper is to investigate the influence of antisymmetric stresses due to relative rotation between the phases. The analytical solutions obtained by series expansions for small and large times (corresponding to the situation of large and small relative velocity between the phases respectively) give a possibility to show the influence and the order of magnitude of the antisymmetric stresses on the motion of each phase. By comparing these solutions with the results for the case when antisymmetric stresses are neglected it is demonstrated that antisymmetric stresses are important whenever there are palpable differences in the rotational motion of each phase. In the present problem this relative rotation is caused by inertia forces which are dominant for small times (large relative velocity). For this case antisymmetric stresses are shown to influence even the zeroth order solution for the fluid velocity tangential to the plate.

2. VOLUME AVERAGED EQUATIONS OF MOTION

In a two-phase continuum theory (two-fluid model) both phases are treated as two mechanically interacting interpenetrating continua. The variables appearing in the balance equations are averaged variables. Hence, fluctuations of the local variables, e.g. the disturbance of the flow field in the vicinity of a particle, are smoothed by the averaging method. However, the statistical properties of these fluctuations are considered in the averaged balance equations (e.g. by the diffusive or Reynolds stresses). Averaged balance equations can be obtained by averaging the balance equations for the local (not averaged) variables. Ishii (1975) applies the method of time averaging, whereas, e.g. Panton (1968), Batchelor (1970), Whitaker (1973), Buyevich & Markov (1975) and Immich (1979) use the method of volume averaging. Drew (1970, 1971) applied an averaging method which is a combination of volume and time averaging.

The volume averaged equations derived by Immich (1980a, b) apply to a situation where the continuous phase is a Newtonian fluid and the dispersed phase is made up of rigid spherical particles. In order that the dispersed phase can be regarded as a continuum, the mean interparticle distance has to be small compared to a macroscopic dimension of the flow field. The volume over which the averaging has to occur must be large enough to include a representative number of particles, but small compared to a macroscopic dimension of the flow field. It has been shown by Immich (1980a, b) that the volume averaged stress tensors of each phase are not symmetric when rotational inertia of the particles is considered and that the volume averaged balance equations for the dispersed phase are closely related to the corresponding equations of a polar fluid (see Eringen 1966 and Cowin 1968).

Usually the volume over which the averaging has to occur is chosen to be spherical (see Drew 1970, 1971, Buyevich & Markov 1975 and Immich 1980a). Provided the statistical properties of the suspension do not vary appreciably over this volume, the volume-averaged variables may be assumed to be equal to averaged variables obtained by an ensemble averaging method (for a discussion see, e.g. Batchelor 1970).

In the present situation of the flow induced by impulsive motion of an infinite flat plate the local (not averaged) flow field of each phase is statistically homogeneous in planes parallel to the flat plate (planes y = constant, see figure 1). Due to strong gradients normal to the plate induced by the shear motion the statistical properties cannot be assumed to be constant over a length l normal to the plate which is of the order of magnitude of some mean interparticle separations. For the present flow problem we choose an averaging volume whose dimensions parallel to the flat plate are large (may go to infinity) to include (or intersect) a great number of particles. However, the dimension normal to the plate is chosen to be only some particle diameters. Hence, for example, the local disturbances of the flow field due to the motion of single particles relative to the fluid are smoothed out in a plane parallel to the plate. Since the dimension of the averaging volume normal to the plate is only some particle diameters, these local disturbances are smoothed out normal to the plate, too. However, provided the boundary layer is large compared to a particle diameter, the shear motion induced by the flat plate is not smoothed out. Formally, the volume averaged equations are the same for a spherical or for the

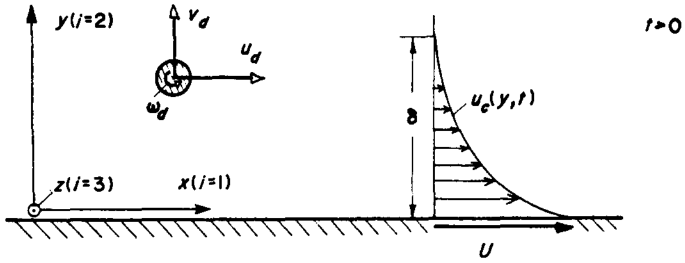

Figure 1. Flow induced by the impulsive motion of an infinite flat plate.

above described averaging volume. However, the transport coefficients in the constitutive equations must take into account the specific flow situation (see discussion of [2.19]).

2.1 *Volume averaged equations*

Heat conduction is not considered here in the problem of the impulsive motion of the flat plate. The materials making up each phase are assumed to be incompressible, the continuous phase being a Newtonian fluid, the dispersed phase consisting of rigid spherical homogeneous particles of equal material density ρ_d and of the same radius a. The volume averaged equations of motion derived by Immich (1980a, b) for the continuous and the dispersed phase are respectively.

Balance of mass

$$\frac{\partial}{\partial t}(1-\alpha)+\frac{\partial}{\partial x_j}[(1-\alpha)v_j^c]=0 \qquad [2.1]$$

$$\frac{\partial \alpha}{\partial t}+\frac{\partial}{\partial x_j}(\alpha v_j^d)=0. \qquad [2.2]$$

Balance of linear momentum

$$(1-\alpha)\rho_c\frac{\mathrm{d}v_i^c}{\mathrm{d}t_c}=\frac{\partial}{\partial x_j}[(1-\alpha)\tau_{ji}^c]+(1-\alpha)\rho_c f_i-F_i \qquad [2.3]$$

$$\alpha\rho_d\frac{\mathrm{d}v_i^d}{\mathrm{d}t_d}=\frac{\partial}{\partial x_j}(\alpha\tau_{ji}^d)+\alpha\rho_d f_i+F_i. \qquad [2.4]$$

Balance of angular momentum

$$(1-\alpha)\epsilon_{ijk}\tau_{jk}^c=M_i^d \qquad [2.5]$$

$$\alpha\rho_d\vartheta_d\frac{\mathrm{d}\omega_i^d}{\mathrm{d}t_d}=\alpha\epsilon_{ijk}\tau_{jk}^d+\frac{\partial}{\partial x_j}(\alpha\mu_{ji}^d)+M_i^d. \qquad [2.6]$$

Here volume averaged variables of the continuous and the dispersed phase are marked by the index c and d, respectively. The average volume fraction of the dispersed phase is α, ρ_c and ρ_d are the material densities of each phase, v_i^c and v_i^d are the averaged velocities, ω_i^d is the average particle rotation which is kinematically independent of v_i^d. The interphase force is F_i and M_i is the interphase moment, i.e. the volume averaged moment on the dispersed phase due to the relative rotational motion of the particles and the ambient fluid. External accelerations (e.g. gravity) are represented by f_i. The stress tensors τ_{ij}^c and τ_{ij}^d include the volume averaged local stresses π_{ij} and the diffusive stresses J_{ij} ("Reynolds stresses") which arise as a result of volume averaging:

$$\tau_{ij}^{c} = \pi_{ij}^{c} - \rho_c J_{ij}^{c} \tag{2.7}$$

$$\tau_{ij}^{d} = \pi_{ij}^{d} - \rho_d J_{ij}^{d} \tag{2.8}$$

where

$$(1-\alpha) J_{ij}^{c} = \langle (1-\tilde{\alpha})(v_i - v_i^{c})(v_j - v_j^{c}) \rangle \tag{2.9}$$

$$\alpha J_{ij}^{d} = \langle \tilde{\alpha}(v_i - v_i^{d})(v_j - v_j^{d}) \rangle . \tag{2.10}$$

Here v_i are local (not averaged) velocities, $\tilde{\alpha}$ is the phase function defined by

$$\tilde{\alpha} = \begin{cases} 1 & \text{in the dispersed phase} \\ 0 & \text{in the continuous phase} \end{cases}$$

and $\langle\,\rangle$ denotes the volume averaging process (see Immich 1980a).

The moment stresses μ_{ij}^{d} arising in the balance of angular momentum of the dispersed phase, [2.6], which are made up of the averaged stress moments and a diffusive angular momentum flux due to the averaging, are not discussed here, since they can be neglected for the case considered in the next section.

The rotational inertia ϑ_d of the particles per mass of a particle is given by

$$\vartheta_d = \frac{2}{5} a^2 , \tag{2.11}$$

where a is the particle radius.

Due to treating both phases as interacting continua there exist velocity vectors v_i^{d} and v_i^{c} at each location of the flow field. The material derivatives of each phase are given by

$$\frac{\mathrm{d}}{\mathrm{d}t_c} = \frac{\partial}{\partial t} + v_j^{c} \frac{\partial}{\partial x_j} \tag{2.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t_d} = \frac{\partial}{\partial t} + v_j^{d} \frac{\partial}{\partial x_j} . \tag{2.13}$$

2.2 *Constitutive equations*

Constitutive equations (rheological equations of state) must be postulated for the interphase force F_i, the interphase moment M_i, the stress tensors τ_{ij}^{c}, τ_{ij}^{d} and the moment stresses μ_{ij}^{d}. In the following we consider laminar flow and we shall restrict to the case of small volume fraction of the dispersed phase, i.e. $\alpha \ll 1$. Hence, the hydrodynamic interaction of the disturbances in the flow field caused by the single particles will be neglected. As a further simplification in the following we use some hypotheses as proposed by Drew & Segel (1971) and Drew (1976).

The hypothesis of phase separation states that a bulk phase variable should depend only on variables from that same phase. Thus, for example, the fluid stress can depend on the fluid shearing $\partial v_i^{c}/\partial x_j$ but not on the shearing $\partial v_i^{d}/\partial x_j$ of the dispersed phase. The interphase force and the interphase moment, however, can depend on variables from both phases.

The volume fraction α, however, can appear in the constitutive equations for either phase, since α is the particle volume fraction, and 1-α is the fluid volume fraction.

The hypothesis of local dependence on dispersed phase variables states that for small volume fraction α nonlocal effects (specifically gradients) of dispersed phase variables do not appear in the constitutive equations of the dispersed phase.

The hypothesis of correct low concentration limits states that when the dispersed phase is sufficiently dilute, the mixture behaves as if it were made up of the continuous phase alone. Moreover, for $\alpha \ll 1$, the particles behave like single particles suspended in the continuous phase.

(a) *Interphase force.* According to the hypothesis of correct low concentration limits we use known results for the force on a single small spherical particle. We consider the pressure force, the drag force, the added mass effect and a lift force due to fluid shear. The volume averaged interphase force is thus given by

$$F_i = p_c \frac{\partial \alpha}{\partial x_i} + \alpha \frac{\rho_d}{\tau_p}(v_i^c - v_i^d) + \alpha \frac{\rho_c}{2}\left(\frac{\mathrm{d}v_i^c}{\mathrm{d}t_c} - \frac{\mathrm{d}v_i^d}{\mathrm{d}t_d}\right) + F_i^L. \quad [2.14]$$

The first expression on the r.h.s. is the volume averaged pressure force (for the derivation see Drew 1971), the second term is the Stokes drag force, τ_P being the relaxation time for particle translation

$$\tau_P = \frac{2}{9}\frac{\rho_d a^2}{\mu}, \quad [2.15a]$$

valid for the particle Reynolds number $\mathrm{Re}_P < 1$

$$\mathrm{Re}_P = \frac{|v_i^d - v_i^c| a \rho_c}{\mu} < 1, \quad [2.15b]$$

μ is the shear viscosity of the fluid material.

The third term is the added mass effect. There have been proposed contradictory forms of the relative acceleration between particles and fluid to be used in the constitutive equation for the added mass effect (see, e.g. Murray 1965, Anderson & Jackson 1967, Drew & Segel 1971, Soo 1977 and Thacker & Lavelle 1978). The form of the relative acceleration we use here coincides with the form applied by Soo (1977) and with an expression proposed by Murray (1965). The same form has been derived analytically by Voinov (1973) for potential flow.

The fourth term on the r.h.s. of [2.14] is the volume averaged lift force considering particle migration across the streamlines in a shear flow. Since there does not exist a general expression for this lift force in a general three dimensional flow, we shall give in the next section an appropriate expression for this lift force for the case of simple shear considered in this paper due to the impulsive motion of the infinite flat plate.

None of the above cited papers takes into account the Basset force which considers the history of the unsteady motion of a single particle (e.g. Landau & Lifshitz 1971). As Anderson & Jackson (1967) point out, in a suspension the historical effect is likely to be erased by the hydrodynamic interactions of the disturbances induced in the vicinity of the particles. In fact, none of the papers mentioned in the introduction about impulsive motion of a suspension considers the Basset force. In the following we neglect the Basset term, too (see Di Giovanni & Lee 1974 and for a discussion of unsteady two-phase equations, see Soo 1977).

(b) *Interphase moment.* Rubinow & Keller (1961) gave an expression for the moment on a rotating sphere in a quiescent fluid which has been extended by Happel & Brenner (1965) for the case of a sphere rotating relative to a rotational fluid. In volume averaged form this interphase moment can be written as (Immich 1979)

$$M_i = \alpha \rho_d \frac{\vartheta_d}{\tau_R}\left(\frac{1}{2}\epsilon_{ijk}\frac{\partial v_k^c}{\partial x_j} - \omega_i^d\right).\dagger \quad [2.16]$$

†Note that for spherical particles there exists no term depending on the relative rotational acceleration, which would correspond to the added mass effect in [2.14] (e.g. Landau & Lifshitz 1971).

Here τ_R is the relaxation time for particle rotation

$$\tau_R = \frac{\rho_d a^2}{15\mu} = 0.3\,\tau_P, \qquad [2.17]$$

where the last expression is given by comparison with [2.15].

(c) *Stress tensor of the continuous phase.* In analogy to Anderson & Jackson (1967) and Drew & Segel (1971) the symmetric part of the volume averaged stress tensor of the continuous phase is written as

$$\tau^c_{(ij)} = \pi^c_{(ij)} - \rho_c J^c_{ij} = -p_c\delta_{ij} + \mu_c(\alpha)\left(\frac{\partial v_i^c}{\partial x_j} + \frac{\partial v_j^c}{\partial x_i}\right) + \lambda_c(\alpha)\frac{\partial v_k^c}{\partial x_k}\delta_{ij}. \qquad [2.18]$$

The dimensions of the averaging volume have been chosen in a way to smooth out only the disturbances of the flow field in the vicinity of the particles, but not to smooth out the gross shear motion induced by the flat plate. For this reason the diffusive stresses $(1-\alpha)\rho_c J^c_{ij} = \langle(1-\tilde{\alpha})\rho(v_i - v_i^c)(v_j - v_j^c)\rangle$ arise to the greatest part due to the above cited disturbances, where $v_i - v_i^c$ is the disturbance due to the motion of the particles relative to the mean velocity v_i^c. The diffusive stresses must be taken into account by the coefficients $\mu_c(\alpha)$ and $\lambda_c(\alpha)$. To a first approximation we may use for the present situation and for the case of $\alpha \ll 1$ the Einstein viscosity (see also Drew 1976).

$$\mu_c(\alpha) = \mu\left(1 + \frac{5}{2}\alpha\right), \qquad [2.19]$$

μ being the viscosity of fluid material.

By postulating that the sum of the averaged normal stresses on a volume element of the continuous phase gives the averaged pressure p_c, the volume viscosity $\lambda_c(\alpha)$ may be written as

$$\lambda_c(\alpha) = -\frac{2}{3}\mu\left(1 + \frac{5}{2}\alpha\right). \qquad [2.20]$$

The balance of angular momentum for the continuous phase, [2.5], shows that the volume averaged stress tensor of the continuous phase is not symmetric. The antisymmetric part of τ^c_{ij} is caused by the interphase moment and may be obtained from [2.5] as

$$\tau^c_{[ij]} = \frac{1}{2}\frac{1}{1-\alpha}\epsilon_{ijk}M_k \qquad [2.21]$$

with M_k given in [2.16].

The total stress tensor τ^c_{ij} is the sum of the symmetric and the antisymmetric part

$$\tau^c_{ij} = \tau^c_{(ij)} + \tau^c_{[ij]}.$$

(d) *Stress tensor of the dispersed phase.* For the case of small volume fraction ($\alpha \ll 1$) the particles do not interact by direct contact (collision). Hence, there is no momentum transport in the dispersed phase by direct contact of the particles. When the hypothesis of local dependence on dispersed phase variables is applied, the averaged stress tensor of the dispersed phase may be shown to be simply

$$\tau^d_{ij} = -p_c\delta_{ij}. \qquad [2.22]$$

As was shown by Panton (1968) and Drew & Segel (1971), the volume averaged (partial) pressure of the dispersed phase may be approximated by the volume averaged pressure of the continuous phase when $\alpha \ll 1$. This result has been considered in [2.22]. The antisymmetric part of τ_{ij}^d which appears in [2.6] may be interpreted as a source of angular momentum of the dispersed phase due to direct transformation of linear momentum into angular momentum by collisional contact of the particles (e.g. Becker & Bürger 1975 and Immich 1980a, b). Hence, antisymmetric stresses in the dispersed phase can be neglected for the case of small volume fraction ($\alpha \ll 1$) considered here.

(e) *Momentum stresses.* The momentum stresses μ_{ij}^d constitute an exchange of angular momentum in the dispersed phase due to direct contact of the particles. For small volume fraction ($\alpha \ll 1$) they therefore may be neglected

$$\mu_{ij}^d = 0 \quad \text{for} \quad \alpha \ll 1 . \qquad [2.23]$$

3. IMPULSIVE MOTION OF THE FLAT PLATE

Consider a flat plate of infinite extent located on the x-axis; the geometry and notation are shown in figure 1.

At time $t < 0$ the suspension occupying the space $y \geq 0$ and the plate are at rest, the pressure p_c and the volume fraction α being constant. At time $t = 0$, the horizontal velocity of the plate is changed impulsively to U. Hence, there are induced horizontal velocities of the continuous and the dispersed phases, u_c and u_d. Transverse velocities v_c and v_d are also induced due to the lateral migration of the particles as a result of the lift force F_i^L. The interphase moment M_i causes the particles to rotate with ω_d (in the z-direction).

3.1 *Lift force* F_i^L

We use an analytical expression for the lift force on a spherical particle derived by Saffman (1965) for the case of a single particle slipping with relative velocity $u_c - u_d$ to a linear shear flow $\mathrm{d}u_c/\mathrm{d}y = \text{const.}$, wall effects being excluded. By taking into account the geometry of figure 1 and under consideration of the direction of this lift force as given by Saffman the lift force is

$$\hat{F}_y{}^L = -6.46\, a^2(\mu\rho_c)^{1/2}(u_c - u_P)\left|\frac{\mathrm{d}u_c}{\mathrm{d}y}\right|^{1/2} . \qquad [3.1]$$

Hence, particles lagging the fluid will be directed towards the wall. The following conditions have to be met for [3.1] to be valid:

$$\mathrm{Re}_P = \frac{a|u_c - u_d|\rho_c}{\mu} \ll 1 \qquad [3.2a]$$

$$\mathrm{Re}_S = \frac{4a^2|\mathrm{d}u_c/\mathrm{d}y|\rho_c}{\mu} \ll 1 \qquad [3.2b]$$

$$\mathrm{Re}_\omega = \frac{4a^2\omega_d\rho_c}{\mu} \ll 1 \qquad [3.2c]$$

and

$$\mathrm{Re}_S \gg \mathrm{Re}_P{}^2 . \qquad [3.2d]$$

In volume averaged form and with [2.15] the lift force is

$$F_y{}^L = -\alpha\frac{6.46}{2\sqrt{2}\pi}\left(\frac{\rho_c\rho_d}{\tau_P}\right)^{1/2}(u_c - u_d)\left|\frac{\mathrm{d}u_c}{\mathrm{d}y}\right|^{1/2} . \qquad [3.3]$$

Rubinow & Keller (1961) gave an analytical expression for the lift force on a spinning sphere moving in a viscous fluid. It can be shown (Di Giovanni & Lee 1974, Otterman 1968 and Immich 1979) that this lift force is an order of magnitude smaller than the Saffman lift force, [3.3], for the problem considered here, hence it is not taken into account.

Though the Saffman lift force is valid only for linear shear in the absence of walls, it is expected to reflect qualitatively the lateral migration of the particles towards the wall in the shear flow considered here. This tendency of the particles which are lagging the fluid to migrate towards the wall is in conformance with a theoretical calculation of the migration velocity of a spherical particle in a laminar shear flow with variable shear rate near a wall of Cox & Hsu (1977) for the case of $\mathrm{Re}_P \ll 1$.

3.2 *Basic equations in dimensionless form*

The following characteristic quantities are used to introduce dimensionless variables:

(a) Particle relaxation time $\tau_P = \dfrac{2}{9}\dfrac{\rho_d a^2}{\mu}$.

(b) Velocity U of the plate.

(c) Velocity V with which the boundary layer grows in the y-direction for a particle-free fluid. We set

$$V = \sqrt{(\nu/\tau_P)} \text{ with } \nu = \mu/\rho_c\,. \tag{3.4}$$

(d) Boundary layer thickness

$$L = \sqrt{(\nu\tau_P)}\,. \tag{3.5}$$

The dimensionless variables are

$$\tilde{u} = \frac{u_c}{U},\quad \tilde{v}_c = \frac{v_c}{\sqrt{(\nu/\tau_P)}},\quad \tilde{y} = \frac{y}{\sqrt{(\nu\tau_P)}},\quad \tilde{p}_c = p_c\frac{\tau_P}{\rho_c\nu} \tag{3.6}$$

$$\tilde{u}_d = \frac{u_d}{U},\quad \tilde{v}_d = \frac{v_d}{\sqrt{(\nu/\tau_P)}},\quad \tilde{\omega}_d = \omega_d\frac{\sqrt{(\nu\tau_P)}}{U},\tau = \frac{t}{\tau_P}\,.$$

Since the interphase force F_i is eliminated when the balance equations of linear momentum of the continuous and the dispersed phases both are added (see [2.3] and [2.4]), it is more convenient to write down the balance of linear momentum of the continuous + dispersed phase and in the following the balance of the dispersed phase alone.

The basic equations in dimensionless form are:

Balance of mass

$$-\frac{\partial\alpha}{\partial\tau} - \tilde{v}_c\frac{\partial\alpha}{\partial\tilde{y}} + (1-\alpha)\frac{\partial\tilde{v}_c}{\partial\tilde{y}} = 0 \tag{3.7}$$

$$\frac{\partial\alpha}{\partial\tau} + \tilde{v}_d\frac{\partial\alpha}{\partial\tilde{y}} + \alpha\frac{\partial\tilde{v}_d}{\partial\tilde{y}} = 0\,. \tag{3.8}$$

Balance of linear momentum of continuous + dispersed phase, x-direction:

$$(1-\alpha)\left(\frac{\partial\tilde{u}_c}{\partial\tau} + \tilde{v}_c\frac{\partial\tilde{u}_c}{\partial\tilde{y}}\right) + \alpha\frac{\rho_d}{\rho_c}\left(\frac{\partial\tilde{u}_d}{\partial\tau} + \tilde{v}_d\frac{\partial\tilde{u}_d}{\partial\tilde{y}}\right) = \left(1 + \frac{3}{2}\alpha + \frac{15}{4}\frac{\vartheta}{a^2}\alpha\right)$$
$$\times\frac{\partial^2\tilde{u}_c}{\partial\tilde{y}^2} + \left(\frac{3}{2} + \frac{15}{4}\frac{\vartheta}{a^2}\right)\frac{\partial\alpha}{\partial\tilde{y}}\frac{\partial\tilde{u}_c}{\partial\tilde{y}} + \frac{15}{2}\frac{\vartheta}{a^2}\left(\alpha\frac{\partial\tilde{\omega}_d}{\partial\tilde{y}} + \frac{\partial\alpha}{\partial\tilde{y}}\tilde{\omega}_d\right), \tag{3.9}$$

y-direction:

$$(1-\alpha)\left(\frac{\partial \tilde{v}_c}{\partial \tau}+\tilde{v}_c\frac{\partial \tilde{v}_c}{\partial \tilde{y}}\right)+\alpha\frac{\rho_d}{\rho_c}\left(\frac{\partial \tilde{v}_d}{\partial \tau}+\tilde{v}_d\frac{\partial \tilde{v}_d}{\partial \tilde{y}}\right)=-\frac{\partial \tilde{p}_c}{\partial \tilde{y}}+\left(1+\frac{3}{2}\alpha\right)\frac{4}{3}\frac{\partial^2 \tilde{v}_c}{\partial \tilde{y}^2}+2\frac{\partial \alpha}{\partial \tilde{y}}\frac{\partial \tilde{v}_c}{\partial \tilde{y}}. \quad [3.10]$$

Balance of linear momentum of dispersed phase, x-direction

$$\left(1+\frac{1}{2}\frac{\rho_c}{\rho_d}\right)\left(\frac{\partial \tilde{u}_d}{\partial \tau}+\tilde{v}_d\frac{\partial \tilde{u}_d}{\partial \tilde{y}}\right)=\tilde{u}_c-\tilde{u}_d+\frac{1}{2}\frac{\rho_c}{\rho_d}\left(\frac{\partial \tilde{u}_c}{\partial \tau}+\tilde{v}_d\frac{\partial \tilde{u}_c}{\partial \tilde{y}}\right), \quad [3.11]$$

y-direction

$$\left(1+\frac{1}{2}\frac{\rho_c}{\rho_d}\right)\left(\frac{\partial \tilde{v}_d}{\rho \tau}+\tilde{v}_d\frac{\partial \tilde{v}_d}{\partial \tilde{y}}\right)=-\frac{\rho_c}{\rho_d}\frac{\partial \tilde{p}_c}{\partial \tilde{y}}+\tilde{v}_c-\tilde{v}_d$$

$$+\frac{1}{2}\frac{\rho_c}{\rho_d}\left[\frac{\partial \tilde{v}_c}{\partial \tau}+\tilde{v}_c\frac{\partial \tilde{v}_c}{\partial \tilde{y}}\right]-C\left(\frac{\rho_d}{\rho_c}\right)^{1/4}\mathrm{Re}_P^{3/2}(\tilde{u}_c-\tilde{u}_d)\left|\frac{\partial \tilde{u}_c}{\partial \tilde{y}}\right|^{1/2}, \quad [3.12a]$$

with

$$C=\frac{6.46\cdot 2^{3/4}}{2\sqrt{2\pi}3^{3/2}}=0.2353 \text{ and } \mathrm{Re}_P=\frac{Ua}{\nu}. \quad [3.12b]$$

Balance of angular momentum of dispersed phase

$$\frac{\partial \tilde{\omega}_d}{\partial \tau}+\tilde{v}_d\frac{\partial \tilde{\omega}_d}{\partial \tilde{y}}+\frac{5}{3}\left(\frac{\partial \tilde{u}_c}{\partial \tilde{y}}+2\tilde{\omega}_d\right)=0. \quad [3.13]$$

Equations [3.7]–[3.13] are seven nonlinear coupled equations for the seven unknown dependent variables α, $\tilde{u}_c$, $\tilde{u}_d$, $\tilde{v}_c$, $\tilde{v}_d$, $\tilde{\omega}_d$, $\tilde{p}_c$.

3.3 *Initial and boundary conditions*

Equations [3.7]–[3.13] are subject to the initial and boundary conditions

$\tau \le 0$: $U = 0$

$$\left.\begin{array}{l}\tilde{u}_c=\tilde{v}_c=\tilde{u}_d=\tilde{v}_d=\tilde{\omega}_d=0\\ \alpha=\text{const.},\ \tilde{p}_c=\text{const.}\end{array}\right\}\ \text{for } \tilde{y}\ge 0 \quad [3.14]$$

$\tau > 0$: $U = \text{const.}$

$$\tilde{u}_c=1,\quad \tilde{v}_c=\tilde{v}_d=0 \quad \text{for } \tilde{y}=0 \quad [3.15]$$

$$\left.\begin{array}{l}\tilde{u}_c=\tilde{v}_c=\tilde{u}_d=\tilde{v}_d=\tilde{\omega}_d=0\\ \alpha=\text{const.},\ \tilde{p}_c=\text{const.}\end{array}\right\}\ \text{for } \tilde{y}\to\infty.$$

Since there are no shear stresses in the dispersed phase (see [2.22]), there can be given no boundary condition for the horizontal velocity of the dispersed phase at the plate. In the same way, there is no boundary condition for the particle rotation at the plate, since momentum stresses in the dispersed phase are neglected. Hence, the particle motion is caused only by the action of the interphase force and the interphase moment as can be seen from [3.11]–[3.13].

The normal velocity of the dispersed phase at the plate can be shown to be zero, since there is no mass flow through the plate:

$$\alpha \rho_d v_d = 0 \text{ at } y = 0 .$$

Since $\alpha \leq \alpha_{\max} < 1$ this gives $v_d = 0$ at $y = 0$.

4. SMALL TIME SOLUTION, $\tau \ll 1$

Approximate solutions to [3.7]–[3.13] are obtained by power series expansions in dimensionless time for small and large times. The following transformation of the independent variables is employed

$$\eta = \frac{\tilde{y}}{2\sqrt{\tau}} = \frac{y}{2\sqrt{(\nu t)}}, \quad \tau = \frac{t}{\tau_P} . \tag{4.1}$$

When the particles are much heavier than the fluid, $\rho_c/\rho_d \ll 1$, the case of $\tau \ll 1$ corresponds to the situation of large particle slip since the particles had not enough time to accomodate their velocity to the fluid velocity due to the action of the Stokes drag force. For particles lighter than the fluid this is not the case since the added mass effect predominates the Stokes drag force, see the factor ρ_c/ρ_d appearing in [3.11] and [3.12]. For small times $\tau \ll 1$, however, we have to restrict to the case of heavy particles, since the continuum description for the suspension requires the boundary layer to include a sufficient number of particles. This restriction can be expressed as

$$\delta \sim \sqrt{(\nu t)} \gg a \tag{4.2}$$

and

$$(\tau/\rho)^{1/2} \gg 1, \ \rho = \rho_c/\rho_d . \tag{4.3}$$

Hence, only for heavy particles the relaxation time is large enough to allow the fluid boundary layer to encompass a significant number of solid particles. An examination of restriction [3.2d], $\mathrm{Re}_S \gg \mathrm{Re}_P^2$ for the Saffman lift force gives

$$t^* \ll \frac{\nu}{U^2} \tag{4.4}$$

when $\mathrm{d}u_c/\mathrm{d}y$ in [3.2b] is approximated by $U/\sqrt{(\nu t)}$. When this is compared with $\tau \ll 1$, one obtains

$$\frac{t^*}{\tau} = \frac{2}{9}\frac{\rho_d}{\rho_c}\left(\frac{aU}{\nu}\right)^2 \ll 1 , \tag{4.5}$$

since $\mathrm{Re}_P = aU/\nu \ll 1$ is required to satisfy [3.2a]. Hence, [4.4] is more restrictive than [4.5] whenever the particles are heavier than the fluid.

For small time, the following series expansions in the small parameter $\tau \ll 1$ for the seven unknown dependent variables α, $\tilde{u}_c$, $\tilde{u}_d$, $\tilde{v}_c$, $\tilde{v}_d$, $\tilde{\omega}_d$, $\tilde{p}_c$ are used (for a comparison see Otterman

1968 and Di Giovanni & Lee 1974):

$$
\begin{aligned}
\tilde{u}_c &= f_0(\eta)+f_1(\eta)\tau^{1/4}+f_2(\eta)\tau^{2/4}+\cdots = f_\gamma(\eta)\tau^{\gamma/4}\\
\tilde{u}_d &= g_0(\eta)+g_1(\eta)\tau^{1/4}+g_2(\eta)\tau^{2/4}+\cdots = g_\gamma(\eta)\tau^{\gamma/4}\\
\tilde{v}_c &= h_0(\eta)+h_1(\eta)\tau^{1/4}+h_2(\eta)\tau^{2/4}+\cdots = h_\gamma(\eta)\tau^{\gamma/4}\\
\tilde{v}_d &= l_0(\eta)+l_1(\eta)\tau^{1/4}+l_2(\eta)\tau^{2/4}+\cdots = l_\gamma(\eta)\tau^{\gamma/4}\\
\tilde{\omega}_d &= m_0(\eta)+m_1(\eta)\tau^{1/4}+m_2(\eta)\tau^{2/4}+\cdots = m_\gamma(\eta)\tau^{\gamma/4}\\
\tilde{p}_c &= p_0(\eta)+p_1(\eta)\tau^{1/4}+p_2(\eta)\tau^{2/4}+\cdots = p_\gamma(\eta)\tau^{\gamma/4}\\
\alpha &= \alpha_0(\eta)+\alpha_1(\eta)\tau^{1/4}+\alpha_2(\eta)\tau^{2/4}+\cdots = \alpha_\gamma(\eta)\tau^{\gamma/4},
\end{aligned}
\qquad [4.6]
$$

where we use the convention that it is to be summed up over repeated Greek indices

$$f_\gamma(\eta)\tau^{\gamma/4} = \sum_{\gamma=0}^{n} f_\gamma(\eta)\tau^{\gamma/4}.$$

The boundary conditions [3.15] for $\tau > 0$ give

$$
\left.\begin{aligned}
&f_0(0)=1\,,\quad f_\gamma(0)=0 \quad \gamma\ge 1\\
&h_\gamma(0)=0 \quad \text{for } \gamma\ge 0\\
&l_\gamma(0)=0 \quad \text{for } \gamma\ge 0
\end{aligned}\right\}\ \text{for } \eta=0 \qquad [4.7]
$$

$$
\left.\begin{aligned}
&f_\gamma=g_\gamma=h_\gamma=l_\gamma=m_\gamma=0 \quad \gamma\ge 0\\
&p_0=\text{const.},\ p_\gamma=0 \quad \gamma\ge 1\\
&\alpha_0=\text{const.},\ \alpha_\gamma=0 \quad \gamma\ge 1\ .
\end{aligned}\right\}\ \text{for } \eta\to\infty.
$$

4.1 *Zeroth order solution*

Use of [4.6] in [3.7]–[3.13], transformed by [4.1], yields perturbation solutions satisfying the boundary conditions [4.7]. The zeroth order solutions are

$$\alpha_0=\text{const.},\quad p_0=\text{const. for } \eta\ge 0 \qquad [4.8]$$

$$f_0(\eta)=1-\frac{2}{\sqrt{\pi}}\int_0^{\beta_I\eta}\exp(-\zeta^2)\,\mathrm{d}\zeta=\operatorname{erfc}(\beta_I\eta)\,, \qquad [4.9a]$$

$$\beta_I^2=\frac{1-\alpha_0+\Lambda\alpha_0}{1+\frac{3}{2}\alpha_0+\frac{15\vartheta}{4a^2}\alpha_0},\quad \Lambda=\frac{1/2}{1+\rho/2}. \qquad [4.9b]$$

$$g_0(\eta)=\rho\Lambda f_0(\eta)=\rho\Lambda\left(1-\frac{2}{\sqrt{\pi}}\int_0^{\beta_I\eta}\exp(-\zeta^2)\,\mathrm{d}\zeta\right). \qquad [4.10]$$

$$h_0=l_0=m_0=0. \qquad [4.11]$$

Here $\rho=\rho_c/\rho_d$ is the density ratio (see [4.3]). To the zeroth order the lift force, [3.3], plays no role,

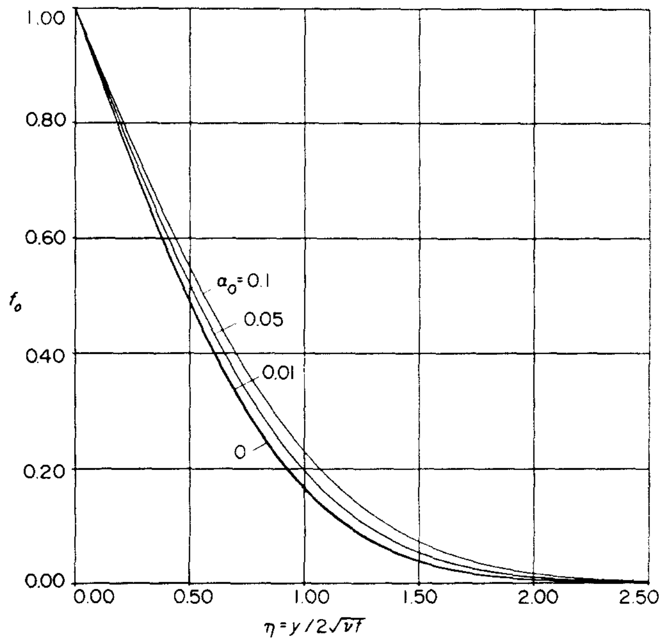

Figure 2. Zeroth order horizontal fluid velocity $f_0 = (u_c/U)_0$, density ratio $\rho = \rho_c/\rho_d = 0.001$. Influence of volume fraction α_0.

hence there is no motion of the particles and the fluid normal to the plate ($l_0 = h_0 = 0$), the constant volume fraction α_0 and the constant pressure p_0 thus being unaffected. Due to their rotational inertia ϑ_d the particles do not rotate ($m_0 = 0$).

The horizontal velocity $f_0 = (u_c/U)_0$ of the continuous phase is shown in figure 2 for various α_0. The case of $\alpha_0 = 0$ corresponds to the classical solution of a pure fluid with $\beta_I = 1$ (see Schlichting 1965). The coefficient β_I, [4.9b] shows the various influences affecting the fluid velocity f_0. The expression $1 + (3/2)\alpha_0$ in the denominator of β_I is a result of the increased viscosity, [2.19] and thus contributes to a greater fluid velocity f_0 than in a pure fluid. Since there are induced local normal velocities in the fluid due to the flow around the single particles as a result of the slip velocity $u_c - u_d > 0$, there is an increased transfer of momentum in the continuous phase, which gives rise to diffusive stresses $\rho_c J^c_{ij}$ in [2.9]. This increased momentum transfer is considered approximately by $\mu_c(\alpha) = (1 + (5/2)\alpha)$, [2.19].

A further increase of the viscosity of the continuous phase is caused by the antisymmetric stresses $\tau^c_{[ij]}$ as can be seen from the expression $(15\vartheta/4a^2)\alpha_0 = (3/2)\alpha_0$ in the denominator of β_I, see also the coefficient of $\partial^2\bar{u}_c/\partial\bar{y}^2$ in [3.9]. In figure 3 the influence of the antisymmetric stresses is shown by comparison with the case of $\tau^c_{[ij]} = 0$, which can be obtained from the above results by setting $\vartheta = 0$. Hence, antisymmetric stresses contribute to a further increase in the zeroth order fluid velocity. Comparison with results of Kline & Allen (1970) show the same effect in the flow of a microstructured fluid induced by the impulsive motion of a flat plate. In their paper antisymmetric stresses are shown to contribute to an increased fluid velocity, too. Comparison with the results of Di Giovanni & Lee (1974) shows that their zeroth order fluid velocity f_0 is smaller than the results obtained here, because they do not take into account an increased viscosity due to the action of diffusive and antisymmetric stresses. Furthermore, their basic equations are in contradiction to the theoretically more founded equations derived by Drew (1970, 1971), Drew & Segel (1971), Buyevich & Markov (1975) and Ishii (1975). Hence, the higher order perturbation solutions obtained by Di Giovanni and Lee cannot be compared with the results here.

The added mass effect, which is characterized by the coefficient Λ appearing in β_I in [4.9b], tends to decrease the fluid velocity f_0, since an increase in β_I induces a decrease in f_0 (see [4.9a]).

The zeroth order particle velocity g_0, [4.10], is shown in figure 4. As can be seen by the coefficient Λ, the added mass effect causes the particle velocity $g_0 = (u_d/U)_0$. Since we consider

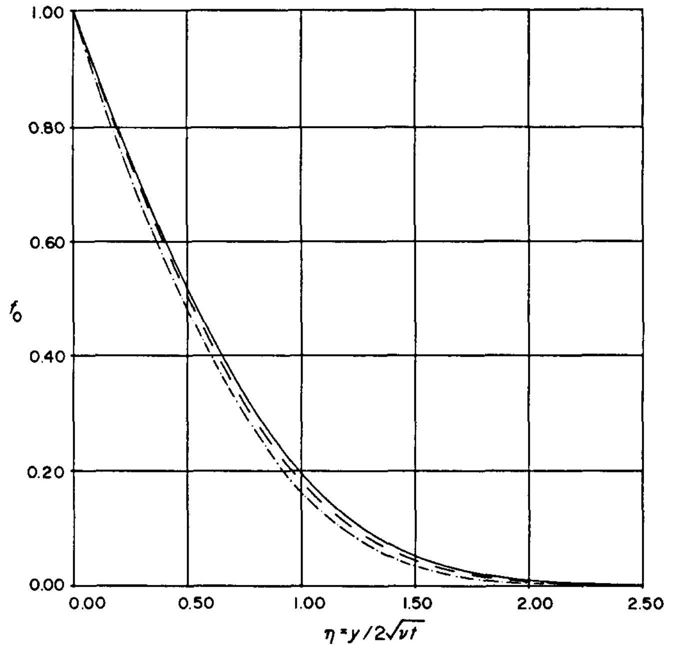

Figure 3. Velocity $f_0 = (u_c/u)_0$; density ratio $\rho = \rho_c/\rho_d = 0.001$; volume fraction $\alpha_0 = 0.05$.—Present theory with $\tau^c_{[ij]} \neq 0$; --- $\tau^c_{[ij]} = 0$; - · · · - Di Giovanni & Lee (1974).

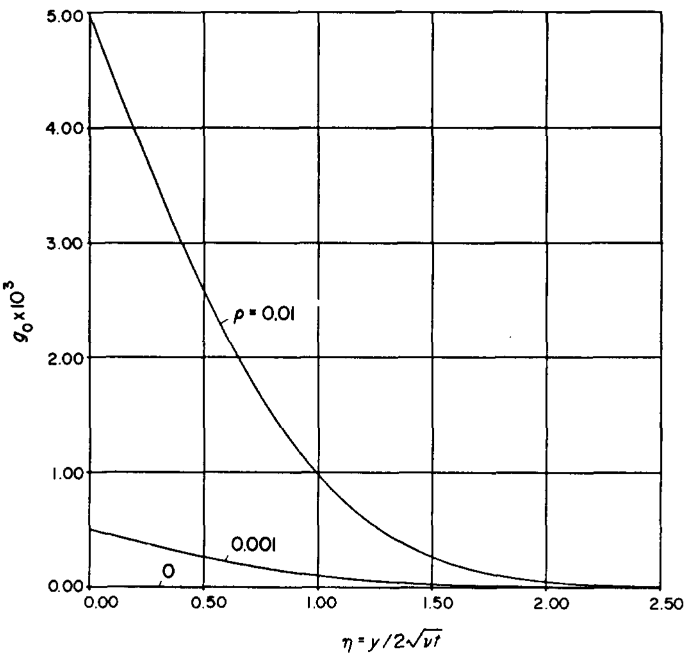

Figure 4. Zeroth order particle velocity $g_0 = (u_d/U)_0$; volume fraction $\alpha_0 = 0.05$. Influence of density ratio $\rho = \rho_c/\rho_d$.

heavy particles, $\rho_c/\rho_d \ll 1$, however, the particles near the plate attain only 5°/∞ of the fluid velocity for $\rho_c/\rho_d = 0.01$ and only 0.5°/∞ for $\rho_c/\rho_d = 0.001$. However, the dominant influence of the density ratio in the added mass effect is demonstrated by figure 4.

4.2 *Higher order solutions*

In the following the nonzero-higher-order perturbation solutions are given.

(a) *Disturbance caused by the interphase moment.* To the zeroth and the first order, the particle rotation is zero ($m_0 = m_1 = 0$). To the second order, the interphase moment M_z causes the particle rotation

$$m_2(\eta) = -\frac{5}{3}\eta\int_\eta^\infty \zeta^{-2} f_0' \,\mathrm{d}\zeta, \tag{4.12a}$$

which gives after some manipulation

$$m_2(\eta) = \frac{10\beta_I}{3\sqrt{\pi}}\exp(-\beta_I^2\eta^2) - \frac{10\beta_I^2}{3}\eta f_0(\beta_I\eta). \tag{4.12b}$$

The dimensionless particle rotation $m_2(\eta)$ is given in figure 5 for various α_0. The greater particle rotation for smaller volume fraction α_0 corresponds to the greater fluid rotation (due to the greater shear rate) for smaller α_0 (see figure 2).

(b) *Disturbance caused by the lift force.* The lift force $F_y{}^L$ induces a third order perturbation velocity l_3 of the particles normal to the plate. The differential equation for $l_3(\eta)$ is

$$\rho\alpha_0\frac{1+\frac{3}{2}\alpha_0}{1-\alpha_0}l_3'' + \frac{3}{2}\eta K l_3' - \frac{9}{4}K l_3 = 3c_L|f_0'|^{1/2}(f_0 - g_0) \tag{4.13a}$$

$$K = 1 - \alpha_0 + \rho\left(\alpha_0 + \frac{1/2}{1-\alpha_0}\right), \quad c_L = \frac{C}{\sqrt{2}}\rho^{-1/4}\,\mathrm{Re}_P{}^{3/2} \tag{4.13b}$$

and C, given in [3.12b].

In figure 6 the numerical solution for l_3/c_L is shown for various volume fractions α_0. Hence, the particle velocity is everywhere negative, the particles are moving normal to the plate. Since $\rho\alpha_0 \ll 1$, [4.13a] may also be solved approximately by applying matched asymptotic expansions, the solution for $\rho\alpha_0 = 0$ representing the outer solution (Immich 1979). The result of the outer solution is shown in figure 6 to coincide for $\eta > 0$ with the solution for $\alpha_0 \leqslant 0.005$. Comparison of

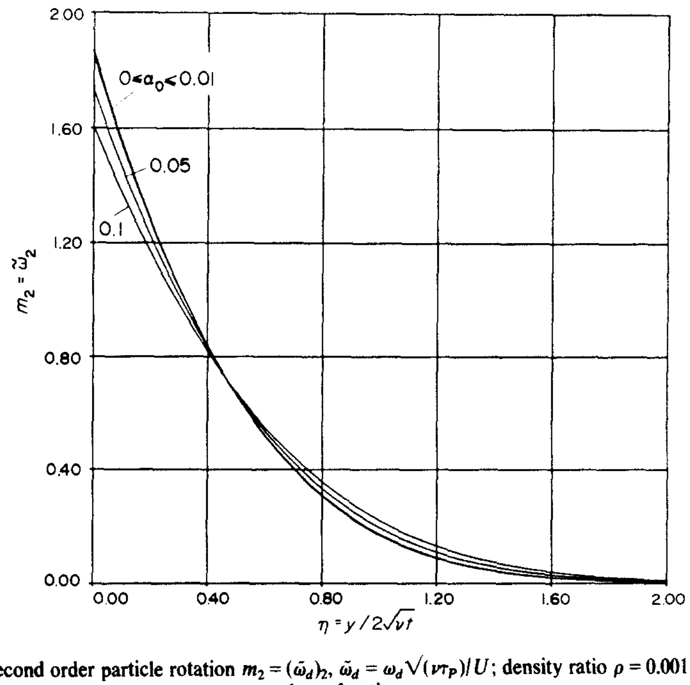

Figure 5. Second order particle rotation $m_2 = (\tilde{\omega}_d)_2$, $\tilde{\omega}_d = \omega_d\sqrt{(\nu\tau_P)}/U$; density ratio $\rho = 0.001$. Influence of volume fraction α_0.

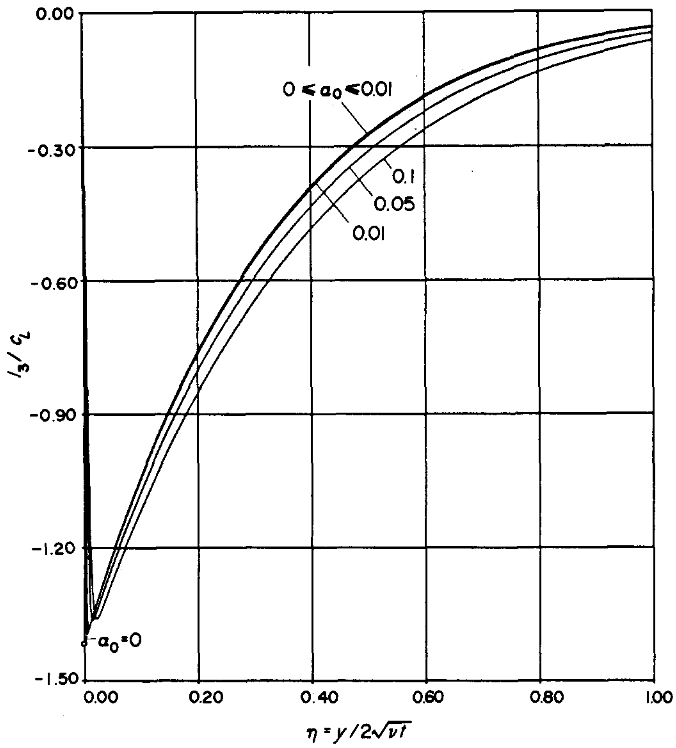

Figure 6. Third order vertical particle velocity $l_3/c_L = (v_d)_3/c_L\sqrt{(v/\tau_P)}$; density ratio $\rho = 0.001$. Influence of α_0.

the above results for l_3 with the case when antisymmetric stresses are neglected shows no essential differences of the two solutions, because f_0 is not affected strongly by setting $\tau^c_{[ij]} = 0$ (see figure 3). Due to the motion of the particles normal to the plate ($l_3 < 0$) a fluid velocity away from the plate ($h_3 > 0$) is induced:

$$h_3(\eta) = -\frac{\alpha_0}{1-\alpha_0} l_3(\eta), \qquad [4.14]$$

the results are shown in figure 7. Note the strong dependence on α_0. The pressure disturbance caused by the lateral migration of the particles is

$$\rho p_1(\eta) = \left(1 + \frac{\rho/2}{1-\alpha_0}\right)\left(\eta l_3 + \frac{5}{2}\int_\eta^\infty l_3(\zeta)\,\mathrm{d}\zeta\right)$$
$$-2c_L(1-\Lambda\rho)\frac{\sqrt{(2\beta_I)}}{\pi^{1/4}}\int_\eta^\infty \exp(-\beta_I^2\zeta^2/2)f_0(\zeta)\,\mathrm{d}\zeta. \qquad [4.15]$$

The results are shown in figure 8. Hence, there is a slight decrease of the pressure near the plate. Since only a pressure drop in direction to the wall can cause the particles to move to the wall, the pressure distribution in figure 8 is explainable.

Due to the migration of the particles to the wall with variable velocity $l_3(\eta)$, a fifth order disturbance of the constant volume fraction ($\alpha_0 = \text{const.}$) is induced:

$$\frac{\alpha_5(\eta)}{\alpha_0} = -\eta^{5/2}\int_\eta^\infty \zeta^{-7/2} l_3'(\zeta)\,\mathrm{d}\zeta \qquad [4.16]$$

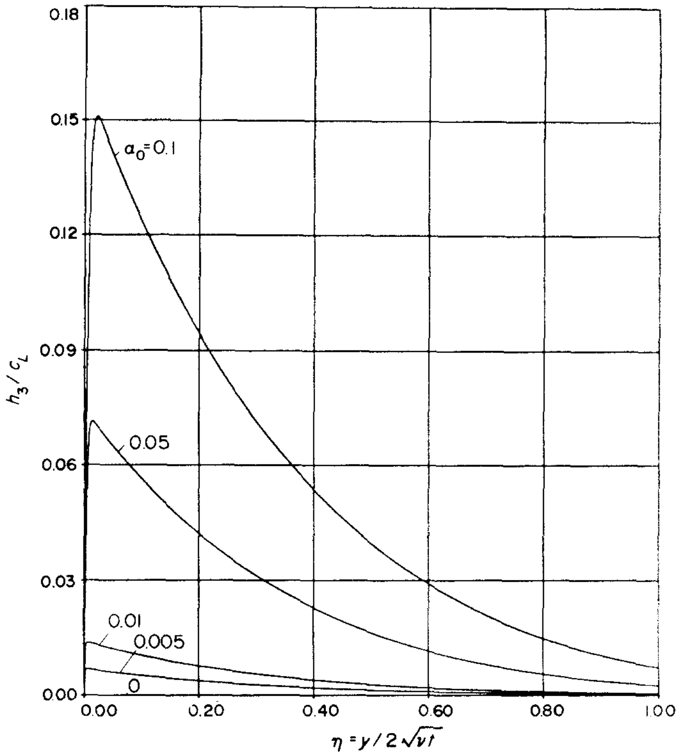

Figure 7. Third order vertical fluid velocity $h_3/c_L = (v_c)_3/c_L\sqrt{(\nu/\tau_P)}$; $\rho = 0.001$. Influence of α_0.

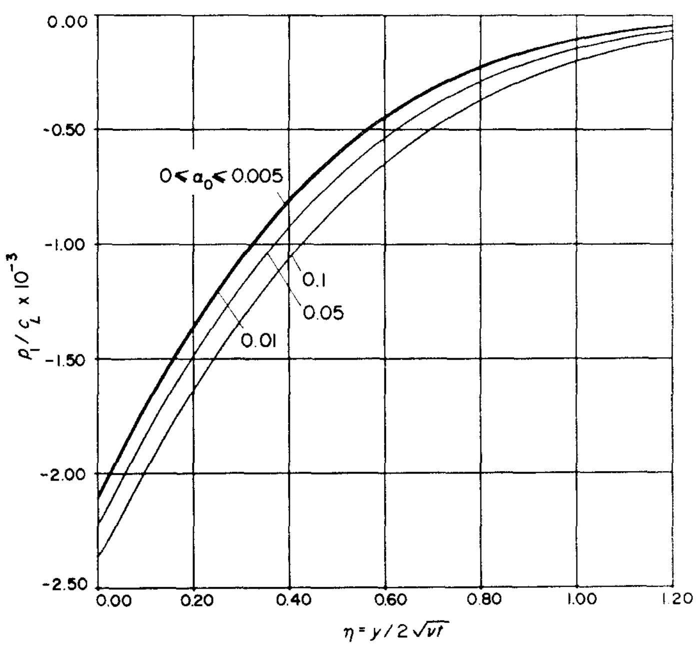

Figure 8. First order pressure disturbance $p_1/c_L = (p_c)_1\tau_p/\rho_c\nu c_L \cdot \rho = 0.001$. Influence of α_0.

where

$$\lim_{\eta \to 0} \frac{\alpha_5}{\alpha_0} = -\frac{2}{5} l_3'(0) . \qquad [4.17]$$

The fifth order disturbance $\alpha_5/\alpha_0 c_L$ is shown in figure 9 indicating the particulate phase tends to increase close to the wall and decrease away from the wall. It must be remembered that in the

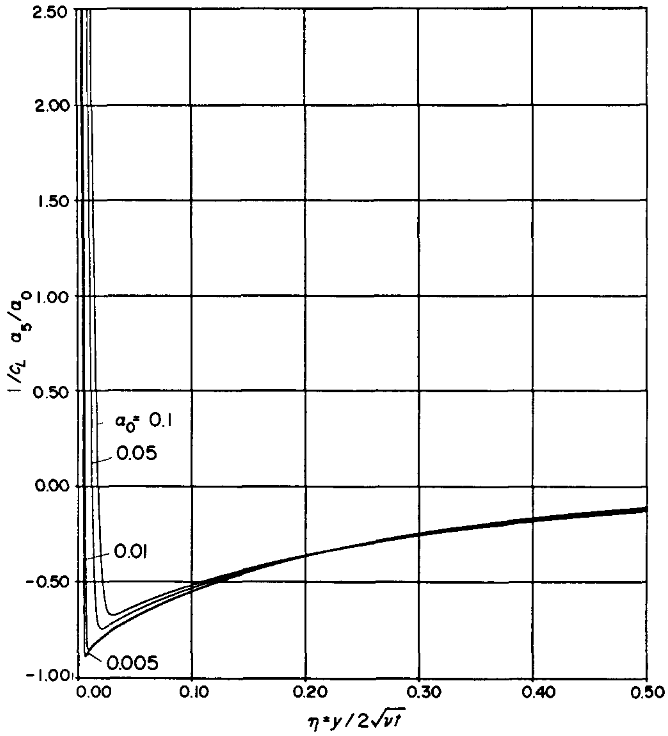

Figure 9. Fifth order disturbance α_5. $\rho = 0.001$. Influence of α_0. $\frac{1}{c_L}\frac{\alpha_5(0)}{\alpha_0} = \begin{cases} 444.6 & \text{for } \alpha_0 = 0.005 \\ 314.4 & \text{for } \alpha_0 = 0.01 \\ 140.6 & \text{for } \alpha_0 = 0.05 \\ 99.4 & \text{for } \alpha_0 = 0.1\ . \end{cases}$

constitutive equations the wall effect has been neglected so that the value of $\alpha_5(0)$ in [4.17] may not be correct. However, since the particles do not interact ($\alpha \ll 1$), the flow behaviour in a distance of some particle radii away from the wall may be expected to be described in the correct way. It can be shown that the volume fraction at the plate increases with time: [4.16] gives

$$\left(\frac{\partial(\alpha/\alpha_0)}{\partial\tau}\right)_{\bar{y}} = \left(\frac{\partial(\alpha/\alpha_0)}{\partial\tau}\right)_{\eta} - \frac{1}{2}\frac{\eta}{\tau}\frac{\partial(\alpha/\alpha_0)}{\partial\eta} = -\frac{1}{2}l_3'(\eta)\tau^{1/4}. \qquad [4.18]$$

Since $l_3'(0) < 0$ (see figure 6), it follows

$$\left(\frac{\partial\alpha}{\partial t}\right)_{y=0} > 0. \qquad [4.19]$$

(c) *Disturbance caused by the Stokes drag force and the interphase moment.* While $f_1 = f_2 = f_3 = 0$, the fourth order horizontal disturbance velocity of the fluid is caused by the Stokes drag force and the interphase moment. The differential equation for f_4 is

$$f_4'' + 2\eta\beta_I^2 f_4' - 4\beta_I^2 f_4 = \frac{4\alpha_0}{\rho}\,\frac{1-\rho\Lambda}{\left(1+\frac{3}{2}\alpha_0+\frac{15}{4}\frac{\vartheta}{a^2}\alpha_0\right)\left(1+\frac{\rho}{2}\right)}f_0 - \frac{15\frac{\vartheta}{a^2}\alpha_0}{1+\frac{3}{2}\alpha_0+\frac{15}{4}\frac{\vartheta}{a^2}\alpha_0}m_2'. \qquad [4.20]$$

The first term on the r.h.s. is the disturbance caused by the Stokes drag force, the second term is the disturbance by the interphase moment. With $m_2(\eta)$ from [4.12b] the analytical solution of [4.20] is

$$f_4(\eta) = -\frac{Q}{2}\left[\frac{1}{\sqrt{\pi}}\beta_I\eta \exp(-\beta_I^2\eta^2) - \beta_I^2\eta^2 f_0(\eta)\right] \quad [4.21a]$$

$$Q = \frac{4\alpha_0}{\rho}\frac{1-\Lambda\rho}{(1-\alpha_0+\Lambda\alpha_0)(1+\rho/2)} + 50\frac{\vartheta}{a^2}\frac{\alpha_0}{1+\frac{3}{2}\alpha_0+\frac{15\vartheta}{4a^2}\alpha_0}. \quad [4.21b]$$

In figure 10, f_4 is given for various α_0. In a similar way as α_0 the density ratio ρ affects the disturbance f_4: the heavier the particles, the greater the negative disturbance f_4 (see the factor α_0/ρ in [4.21b]).

Comparison with the case when the antisymmetric stresses $\tau^c_{[ij]}$ are neglected shows appreciable differences in the fourth order disturbance of the fluid velocity as shown in figure 11. The very much greater disturbance velocity f_4 when $\tau^c_{[ij]}$ is considered reflects the matter of fact that a part of the momentum of the continuous phase is needed to cause the particle rotation $m_2 \neq 0$.

The fourth order particle velocity, caused by the Stokes drag force and by the added mass effect resulting from the fourth order slip velocity is

$$g_4(\eta) = \rho\Lambda f_4(\eta) + 2\frac{1-\Lambda\rho}{1+\rho/2}\left[\frac{1}{2}f_0(\eta) - \frac{\beta_I}{\sqrt{\pi}}\eta\exp(-\beta_I^2\eta^2) + \beta_I^2\eta^2 f_0(\eta)\right]. \quad [4.22]$$

In figure 12 $g_4(\eta)$ obtained from [4.22] is compared with the case when $\tau^c_{[ij]} = 0$. As $f_0(\eta)$ does not vary essentially when $\tau^c_{[ij]} = 0$, there are no essential differences.

With the above results, the balance of angular momentum of the dispersed phase, [3.13], gives an opportunity to calculate the sixth order particle rotation ($m_3 = m_4 = m_5 = 0$):

$$m_6(\eta) = -\frac{5}{3}\eta^3\int_\eta^\infty \zeta^{-4}(f_4' + 4m_2)\,d\zeta. \quad [4.23]$$

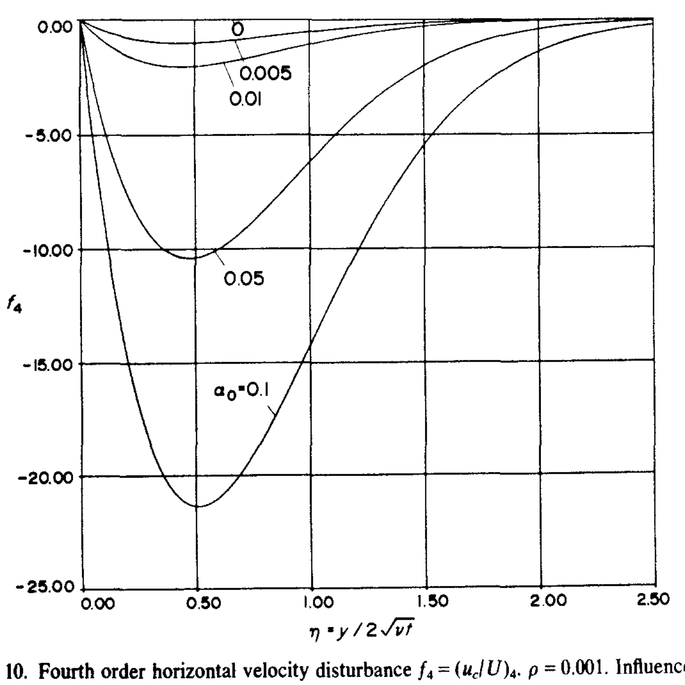

Figure 10. Fourth order horizontal velocity disturbance $f_4 = (u_c/U)_4$. $\rho = 0.001$. Influence of α_0.

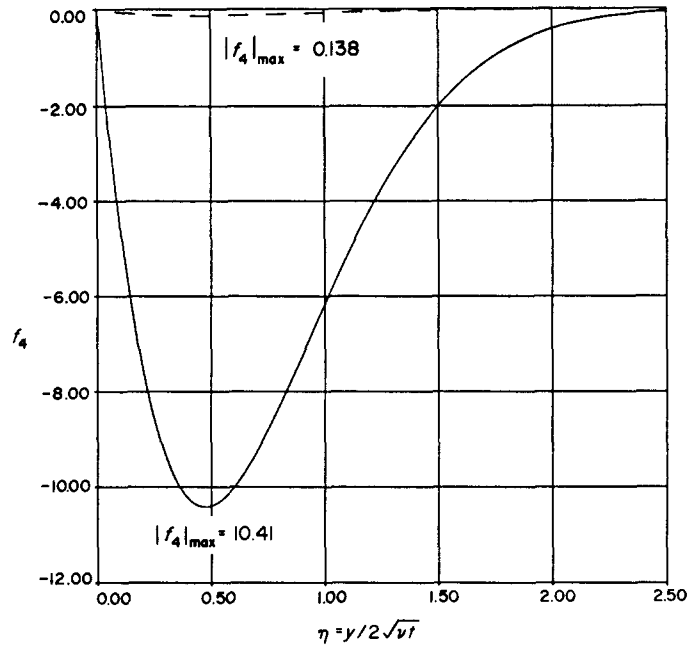

Figure 11. Velocity $f_4 = (u_c/U)_4$. $\rho = 0.001$. $\alpha_0 = 0.05$. — present theory with $\tau^c_{[ij]} \neq 0$. - - - $\tau^c_{[ij]} = 0$.

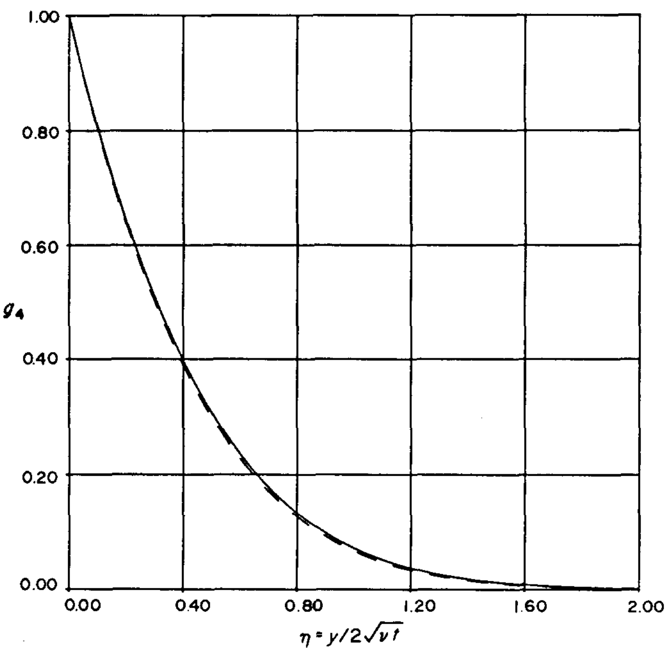

Figure 12. Particle velocity $g_4 = (u_d/U)_4$. $\rho = 0.001$. $\alpha_0 = 0.05$. —Present theory with $\tau^c_{[ij]} \neq 0$; - - - $\tau^c_{[ij]} = 0$.

Hence, m_6 is proportional to the difference between the fourth order fluid rotation and the second order particle rotation. With equations [4.12b] and [4.21] the analytical solution of [4.23] is

$$m_6(\eta) = \frac{5}{3}\beta_I\left\{\left[\left(\frac{20}{3} - \frac{Q}{2}\right)\beta_I\eta + \left(\frac{40}{9} - \frac{2}{3}Q\right)\beta_I^3\eta^3\right]f_0(\eta) - \frac{1}{\sqrt{\pi}}\left[\frac{40}{9} - \frac{Q}{6} + \left(\frac{40}{9} - \frac{2}{3}Q\right)\beta_I^2\eta^2\right]\exp\left(-\beta_I^2\eta^2\right)\right\}, \qquad [4.24]$$

with Q given in [4.21b]. The result is shown in figure 13 for various α_0.

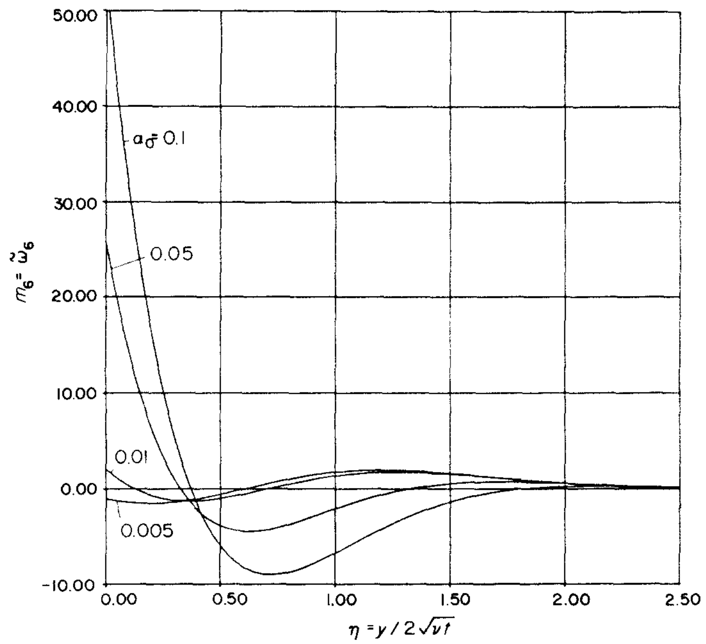

Figure 13. Sixth order particle rotation. $\rho = 0.001$. Influence of volume fraction.

4.3 *Numerical example*

For small times, $\tau \ll 1$, we have to consider an example where the particle relaxation time τ_P is large enough for the small time solution $\tau = t/\tau_P \ll 1$ to admit physically meaningful times t. Hence we consider a gas-particulate suspension with the following parameters

$$\rho_c = 1\ \mathrm{kg/m^3}\,, \quad \rho_d = 10^3\ \mathrm{kg/m^3}\,, \quad \mu = 1.7 \cdot 10^5\ \mathrm{kg/ms}$$

$$a = 10^{-3}\ \mathrm{m}\,, \quad \tau_P = (2/9)\rho_d a^2/\mu = 13.07\ \mathrm{s}\,.$$

As can be seen from figure 11, the greatest disturbance of the fluid horizontal velocity occurs at $\eta \approx 0.5$. For $\alpha_0 = 0.05$ we obtain the following relation for $\eta = 0.5$ (see [4.25]): $u_c/U = 0.52 - 10.41\tau$ for $\eta = 0.5$. For $\tau = 0.02$ (time $t = 0.26$ s) we obtain $u_c/U = 0.31$ at the position $\eta = 0.5$, $y = \eta 2\sqrt{(\nu t)} = 2.1 \cdot 10^{-3}$m.

When this is compared with the case of a particle-free fluid ($\alpha_0 = 0$) (see figure 2), we see that at $\eta = 0.5$ the fluid velocity u_c is reduced by 36 per cent due to the presence of the particles (see discussion of [4.21]).

Comparison of the above with the case when antisymmetric stresses are neglected gives

$$u_c/U = 0.520 - 0.14\tau = 0.517 \quad \text{at } \eta = 0.5 \ (\tau^c_{[ij]} = 0),$$

which demonstrates again the strong influence of antisymmetric stresses on u_c.

Since the disturbance $f_4(\eta)$ may reach rather high values (see figure 10) we conclude from [4.21], that the order of magnitude of the ratio α_0/ρ may not be much greater than about 10 (in the present example $\alpha_0/\rho = 50$ may constitute an upper limit). For greater values of α_0/ρ the convergence of the series expansion $u_c/U = f_0 + f_4\tau + \cdots$ would require unreasonably small values of $\tau \ll 1$. For values of $\alpha_0/\rho \gg 10$ the influence of the dispersed phase in the suspension is overwhelming (see factor $\alpha\rho_d/\rho_c = \alpha/\rho$ in [3.9]). For this case the series expansion [4.6] is not appropriate, since the disturbances induced by each phase are not of a comparable order of magnitude.

The particle horizontal velocity attains its greatest values at $\eta = 0$ (see figures 4 and 12). For the present example we have

$$u_d/U = 0.0005 + 0.9990\tau = 0.0205, \quad \text{for } \eta = 0, \quad \tau = 0.02\,,$$

indicating that for heavy particles ($\rho \ll 1$) the disturbances by the Stokes drag force (term 0.9990τ) is an order of magnitude greater than the added mass effect (term 0.0005).

Comparison of the particle rotation $\tilde{\omega}_d$ with the fluid rotation $\tilde{\omega}_c = -(1/2)\partial\tilde{u}_c/\partial\tilde{y} = -(1/4)f'_\gamma\tau^{(\gamma-2)/4}$ gives

$$\frac{\tilde{\omega}_d}{\tilde{\omega}_c} = -\frac{\omega_d}{\dfrac{1}{2}\dfrac{\partial u_c}{\partial y}} = -4\frac{m_\gamma}{f'_\delta}\tau^{(\gamma-\delta+2)/4}\,.$$

To the lowest order for the present example we have

$$\frac{\tilde{\omega}_d}{\tilde{\omega}_c} = -4\frac{m_2}{f'_0} = 6.66\tau = 0.13 \quad \text{for } \eta = 0,\ \tau = 0.02.$$

Hence the particle rotation close to the wall attains 13 per cent of the fluid rotation.

For a discussion of the disturbances v_c, v_d, p_c and α, all of which are caused by the lift force, we have to restrict to rather low flat plate velocities U, since restrictions [3.2] have to be met.

For $U = 0.01$ m/s and for $\tau = 0.02$ at $y = 0$ we have

a[m]	τ_P[s]	Re_P	$\lvert du_c/dy\rvert$[s^{-1}]	Re_S	$\mathrm{Re}_S/\mathrm{Re}_P^2$
10^{-3}	13.07	0.588	2.46	0.580	1.68
10^{-4}	0.13	0.059	24.64	0.058	16.80

where

$$\frac{du_c}{dy} = -\frac{1}{2}f'_0\frac{U}{\sqrt{(\nu\tau_P)}}\tau^{-1/2} = \frac{\beta_I}{\sqrt{\pi}}\frac{U}{\sqrt{(\nu\tau_P)}}\tau^{-1/2} \text{ at } y = 0.$$

The following tabulated results are valid at the wall ($y, \eta \equiv 0$) for α and p_c or in the immediate vicinity of the wall for v_c and v_d. The results for $a = 10^{-3}$ m are expected to be valid only qualitatively, since conditions [3.2] are not met for this case. For $\tau = 0.02$ and $p_\infty = 1$ bar we have

a[m]	v_d/U ($\eta \approx 0$)	v_c/U ($\eta \approx 0$)	$\alpha/\alpha_0 = 1 + (\alpha_5/\alpha_0)\tau^{5/4}$	$p_c/p_\infty = 1 + (p_1/\bar{P}_\infty)\tau^{1/4}$
10^{-3}	-3.6×10^{-3}	1.9×10^{-4}	1.446	4.5×10^{-9}
10^{-4}	-1.1×10^{-3}	6.0×10^{-5}	1.014	1.4×10^{-8}

Hence the only palpable effect of the lift force is an increase in the volume fraction near the wall of about 45 per cent for $a = 10^{-3}$ and of 1.4 per cent for $a = 10^{-4}$ [m]. The other disturbances v_d, v_c and the pressure disturbances are extremely small.

4.4 *Summary of results for $\tau \ll 1$*

The series expansions [4.6] are given as far as they have been examined by the author, where the terms in brackets are the next nonzero terms not considered here.

$$\tilde{u}_c = f_0(\eta) + f_4(\eta)\tau^{4/4} + [f_5(\eta)\tau^{5/4}] + \cdots$$

$$\tilde{u}_d = g_0(\eta) + g_4(\eta)\tau^{4/4} + [g_5(\eta)\tau^{5/4}] + \cdots$$
$$\tilde{v}_c = h_3(\eta)\tau^{3/4} + [h_5(\eta)\tau^{5/4}] + \cdots$$
$$\tilde{v}_d = l_3(\eta)\tau^{3/4} + [l_5(\eta)\tau^{5/4}] + \cdots \qquad [4.25]$$
$$\tilde{\omega}_d = m_2(\eta)\tau^{2/4} + m_6(\eta)\tau^{6/4} + \cdots$$
$$\tilde{p}_c = p_0 + p_1(\eta)\tau^{1/4} + [p_3(\eta)\tau^{3/4}] + \cdots$$
$$\alpha = \alpha_0 + \alpha_5(\eta)\tau^{5/4} + \cdots.$$

5. LARGE TIME SOLUTION, $\tau \gg 1$

For times large compared with the particle relaxation time, only very small slip between particles and fluid is to be expected. The boundary layer now in every case is large enough to encompass a significant number of particles, hence the heavy particle restriction [4.3] is not necessary.

An examination of restriction [3.2d], $\mathrm{Re}_S \gg \mathrm{Re}_P{}^2$ for large times, $\tau \gg 1$, gives

$$\frac{\mathrm{Re}_S}{\mathrm{Re}_P{}^2} = 0\left(\frac{\sqrt{\nu}}{U\sqrt{t}}\frac{U^2}{|u_c - u_d|^2}\right) \gg 1, \qquad [5.4]$$

since $|u_c - u_d|^2 \to 0$ for $\tau \gg 1$. In a similar way, it can be shown that $\mathrm{Re}_P \ll 1$ and $\mathrm{Re}_S \ll 1$, hence, conditions [3.2a, b] and also condition [3.2c] are met, since the particle rotation is smaller or equal to the fluid rotation in the problem considered here.

The following series expansions are used:

$$\tilde{u}_c = F_0(\eta) + F_1(\eta)\tau^{-1/4} + \cdots = F_\gamma(\eta)\tau^{-\gamma/4}$$
$$\tilde{u}_c - \tilde{u}_d = G_0(\eta) + G_1(\eta)\tau^{-1/4} + \cdots = G_\gamma(\eta)\tau^{-\gamma/4}$$
$$\tilde{v}_c = H_0(\eta) + H_1(\eta)\tau^{-1/4} + \cdots = H_\gamma(\eta)^{-\gamma/4}$$
$$\tilde{v}_c - \tilde{v}_d = L_0(\eta) + L_1(\eta)\tau^{-1/4} + \cdots = L_\gamma(\eta)\tau^{-\gamma/4} \qquad [5.1]$$
$$\tilde{\omega}_c - \tilde{\omega}_d = M_0(\eta) + M_1(\eta)\tau^{-1/4} + \cdots = M_\gamma(\eta)\tau^{-\gamma/4}$$
$$\tilde{p}_c = p_0(\eta) + p_1(\eta)\tau^{-1/4} + \cdots = p_\gamma(\eta)\tau^{-\gamma/4}$$
$$\alpha = \alpha_0(\eta) + \alpha_1(\eta)\tau^{-1/4} + \cdots = \alpha_\gamma(\eta)\tau^{-\gamma/4},$$

where $\tilde{\omega}_c$ is the dimensionless fluid rotation given by

$$\tilde{\omega}_c = -\frac{1}{2}\frac{\partial \tilde{u}_c}{\partial \tilde{y}} = -\frac{1}{4}\tau^{-1/2}\frac{\partial \tilde{u}_c}{\partial \eta} = -\frac{1}{4}F'_\gamma\tau^{-(\gamma+2)/4}. \qquad [5.2]$$

The particle rotation $\tilde{\omega}_c$ obtained from the above is

$$\tilde{\omega}_d = \tilde{\omega}_c - M_\gamma(\eta)\tau^{-\gamma/4} = -\frac{1}{4}F'_\gamma\tau^{-(\gamma+2)/4} - M_\gamma(\eta)\tau^{-\gamma/4}. \qquad [5.3]$$

The boundary conditions [3.15] for $\tau > 0$ are

$$\left.\begin{array}{lll} F_0(0) = 1, \quad F_\gamma(0) = 0 & & \gamma \geq 1 \\ H_\gamma(0) = 0 & & \gamma \geq 0 \\ L_\gamma(0) = 0 & & \gamma \geq 0 \end{array}\right\} \text{ for } \eta = 0 \qquad [5.5]$$

$$\left.\begin{array}{lll} F_\gamma = G_\gamma = H_\gamma = L_\gamma = M_\gamma = 0 & \gamma \geq 0 \\ p_0 = \text{const.},\ p_\gamma = 0 & \gamma \geq 1 \\ \alpha_0 = \text{const.},\ \alpha_\gamma = 0 & \gamma \geq 1 \end{array}\right\} \quad \text{for } \eta \to \infty.$$

5.1 *Zeroth order solution*

The zeroth order solutions are

$$\alpha_0 = \text{const.},\ p_0 = \text{const.},\ G_0 = H_0 = L_0 = M_0 = 0,$$

$$F_0(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\beta_{II}\eta} \exp(-\zeta^2)\,\mathrm{d}\zeta = \operatorname{erfc}(\beta_{II}\eta) \qquad [5.6a]$$

with

$$\beta_{II}^2 = \frac{1 - \alpha_0 + \alpha_0/\rho}{1 + \frac{3}{2}\alpha_0}. \qquad [5.6b]$$

Hence, we obtain the expected result that to the zeroth order there is no slip velocity ($G_0 = 0$) and no rotation of the particles relative to the fluid ($M_0 = 0$). The zeroth order fluid velocity (which is identical with the zeroth order particle velocity ($G_0 = 0 = (\bar{u}_c - \bar{u}_d)_0$) is shown in figure 14 for various density ratios $\rho = \rho_c/\rho_d$. Since heavy particles withdraw more momentum from the fluid until the velocities are balanced ($G_0 = 0$), the boundary layer for heavy particles is thinner than for light particles.

For comparison the results obtained by Di Giovanni and Lee are given too, the only difference in their zeroth order solution being the missing of the term $(3/2)\,\alpha_0$ in β_{II}^2 in [5.6b] due to the neglection of an increased viscosity resulting from diffusive stresses, [2.19]. Note the missing of the added mass effect (no coefficient Λ) and of antisymmetric stresses (no coefficient ϑ) in β_{II}^2, since there is no zeroth order relative acceleration and no relative rotation between

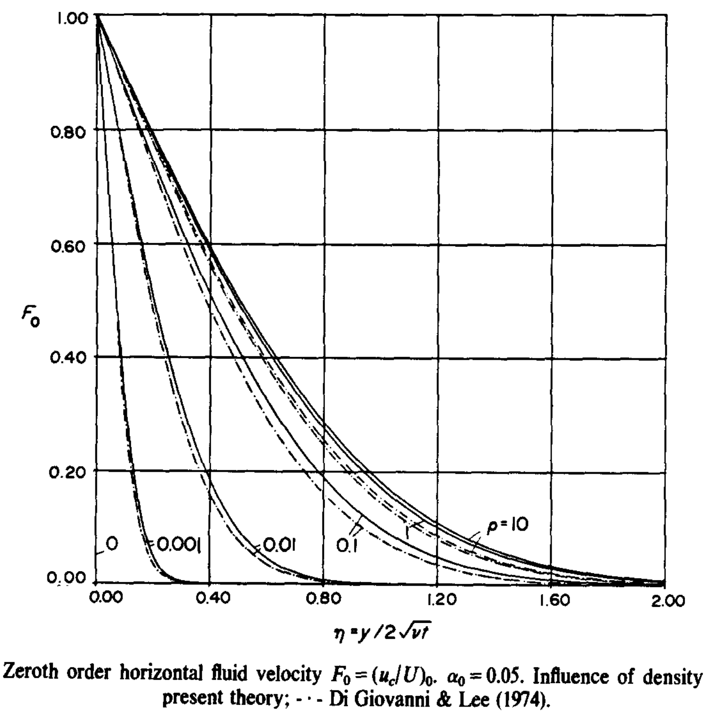

Figure 14. Zeroth order horizontal fluid velocity $F_0 = (u_c/U)_0$. $\alpha_0 = 0.05$. Influence of density ratio. —— present theory; - · - Di Giovanni & Lee (1974).

the phases. In a similar way as the density ratio an increase in volume fraction α_0 causes a greater disturbance and thus a smaller velocity $(F_0)_{\eta=\text{const}}$ when $\rho = \rho_c/\rho_d < 0.4$. When $\rho > 0.4$, an increase in volume fraction α_0 causes a greater velocity $(F_0)_{\eta=\text{const.}}$, since $\beta_{II} \leq 1$ for $\rho \geq 0.4$. Hence, the increased continuous phase viscosity dominates the disturbance caused by the momentum which was withdrawn from the fluid to accelerate the particles to the fluid velocity for $\rho \geq 0.4$.

5.2 *Higher order solutions*

(a) *Solutions for the slip velocities* G_4, H_5 and L_5. A fourth order slip velocity due to the Stokes drag force is given by ($G_0 = G_1 = G_2 = G_3 = 0$):

$$G_4(\eta) = -\frac{\eta}{2}F_0' = \frac{\beta_{II}}{\sqrt{\pi}}\eta \exp(-\beta_{II}^2\eta^2) \geq 0\,, \qquad [5.7]$$

which is shown in figure 15 for various density ratios $\rho = \rho_c/\rho_d$. For $\rho < 0.4$, an increase of α_0 corresponds qualitatively to a decrease of ρ in figure 15, for $\rho > 0.4$ the increased fluid velocity dominates as was the case for F_0. For comparison, for $\rho = \rho_c/\rho_d = 10$, the results are shown in figure 16.

Since the particles are slipping ($G_4 \geq 0$) in the shear field, the Saffman lift force causes a fifth order slip velocity normal to the plate

$$L_5(\eta) = c_L|F_0'|^{1/2}G_4 = \frac{c_L\sqrt{2}}{\pi^{3/4}}\beta_{II}^{3/2}\,\eta \exp\left(-\frac{3}{2}\beta_{II}^2\eta^2\right) \geq 0. \qquad [5.8]$$

The result for $L_5(\eta)$ is shown in figure 17 for various density ratios $\rho = \rho_c/\rho_d$. The fifth order particle velocity can be obtained from L_5 to be

$$(\tilde{v}_d)_5 = -(1-\alpha_0)L_5(\eta) \leq 0\,, \qquad [5.9]$$

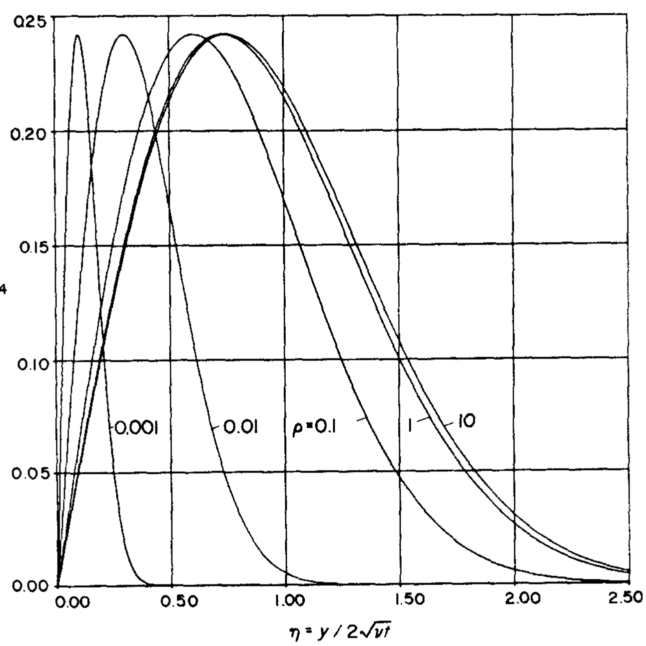

Figure 15. Fourth order slip velocity $G_4 = (u_c/U - u_d/U)_4$. $\alpha_0 = 0.05$. Influence of ρ.

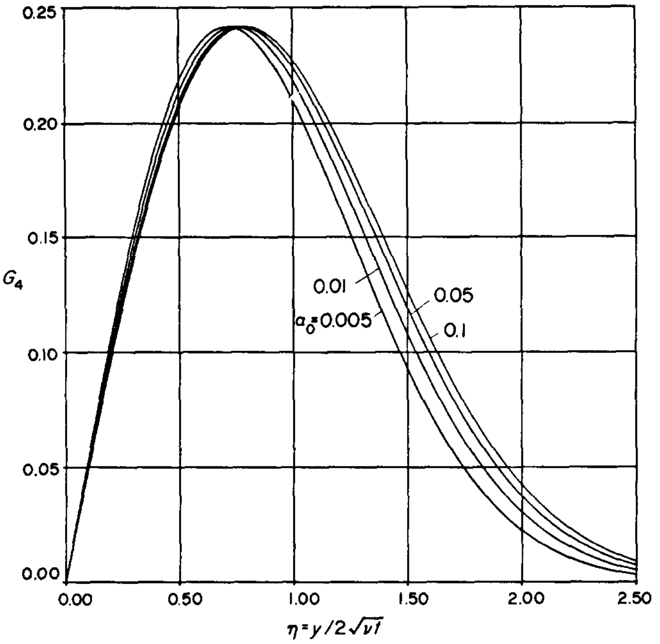

Figure 16. Slip velocity G_4. $\rho = 10$. Influence of α_0.

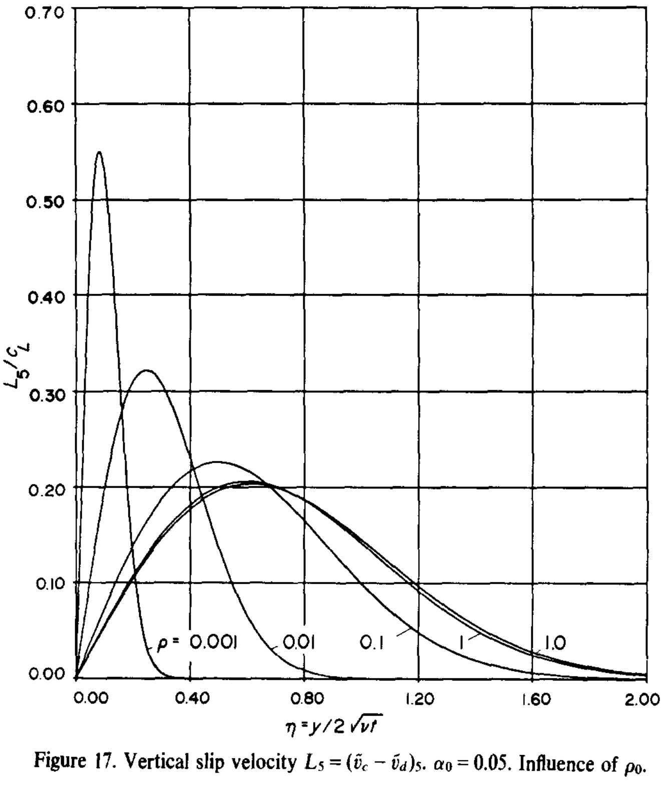

Figure 17. Vertical slip velocity $L_5 = (\bar{v}_c - \bar{v}_d)_5$. $\alpha_0 = 0.05$. Influence of ρ_0.

when the result of the balance of mass is considered, which gives

$$\alpha_0 L_5(\eta) = H_5(\eta) = (\bar{v}_c)_5 \geq 0 \,. \qquad [5.10]$$

As in the case of $\tau \ll 1$, the particles are moving to the plate for $\tau \gg 1$, since $(\bar{v}_d)_5 \leq 0$, [5.9]. The fifth order fluid velocity, given in [5.10], is directed away from the plate.

(b) *Third order solutions for* α_3 *and* F_3. The disturbance of the volume fraction, caused by the lateral migration of the particles is obtained from the balance of mass to be

$$\frac{3}{2}\frac{\alpha_3}{\alpha_0}+\eta\frac{\alpha_3'}{\alpha_0}=-(1-\alpha_0)L_5'. \quad [5.11]$$

With the result of [5.8] we obtain

$$\frac{\alpha_3(\eta)}{\alpha_0}=-(1-\alpha_0)c_L\frac{\sqrt{2}\beta_{II}^{3/2}}{\pi^{3/4}}\left[\left(\frac{2}{3}-\frac{6}{7}\beta_{II}^2\eta^2\right)\exp\left(-\frac{3}{2}\beta_{II}^2\eta^2\right)+3\beta_{II}^2\,\eta^{-3/2}\int_0^\eta\zeta^{5/2}\right.$$
$$\left.\times\left(\frac{2}{3}-\frac{6}{7}\beta_{II}^2\zeta^2\right)\exp\left(-\frac{3}{2}\beta_{II}^2\zeta^2\right)\mathrm{d}\zeta\right]. \quad [5.12]$$

The boundary value at the plate is

$$\frac{\alpha_3(0)}{\alpha_0}=-(1-\alpha_0)c_L\frac{2\sqrt{2}}{3\pi^{3/4}}\beta_{II}^{3/2}. \quad [5.13]$$

The results are shown in figure 18. The instantaneous decrease of α near the plate results from the migration of the particles to the wall with different velocity ($L_5'(0)>0$). However, it can be shown that the volume fraction at the plate increases with time as was the case for $\tau \ll 1$. From [5.11] and [5.12] one obtains

$$\left(\frac{\partial(\alpha/\alpha_0)}{\partial\tau}\right)_{\bar{y}}=+\frac{1}{2}\beta_{II}(1-\alpha_0)L_5'(\eta)\tau^{-7/4}>0 \text{ for } \eta=0. \quad [5.14]$$

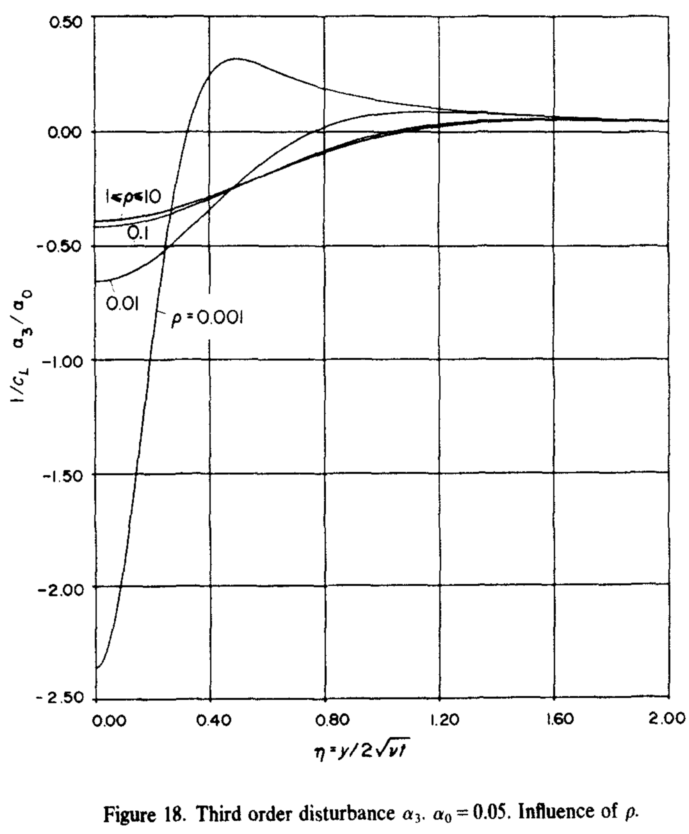

Figure 18. Third order disturbance α_3. $\alpha_0=0.05$. Influence of ρ.

Some part of the fluid momentum is consumed to cause the lateral migration of the particles and the change of the constant volume fraction α, which gives rise to the negative disturbance velocity F_3:

$$F_3''+2\eta^*F_3'+3F_3=c_L Q_{F3}\exp(-\eta^{*2})\Big[\Big(\frac{1}{3}P\eta^*+\frac{39}{14}\eta^*+\frac{6}{7}P\eta^{*3}\Big)\exp\Big(-\frac{3}{2}\eta^{*2}\Big)$$

$$-3\Big(P\eta^{*-1/2}-\frac{9}{8}\eta^{*-5/2}\Big)\int_0^{\eta^*}\zeta^{5/2}\Big(\frac{2}{3}-\frac{6}{7}\zeta^2\Big)\exp\Big(-\frac{3}{2}\zeta^2\Big)\,\mathrm{d}\zeta\Big] \qquad [5.15]$$

with

$$\eta^*=\beta_{II}\eta,\quad Q_{F3}=\frac{4\sqrt{2}\beta_{II}^{3/2}}{\pi^{5/4}}\frac{1-\alpha_0}{1+\frac{3}{2}\alpha_0}\alpha_0,\quad P=\frac{1-(5/2)\rho}{(1-\alpha_0)\rho+\alpha_0}. \qquad [5.16]$$

The results of the numerical solution are shown in figure 19. Hence, heavy particles cause a greater disturbance than light particles. In the case of $\tau \ll 1$, the lift force caused a fifth order disturbance velocity $f_5(\eta)$, which is not given here.

(c) *Solutions for* F_4 *and* M_6. In the contrary to the small time solution, where antisymmetric stresses already affect the zeroth order fluid velocity, in the large time solution antisymmetric stresses appear for the first time in the solutions for F_4 and M_6.

The differential equation for F_4 is

$$\Big(1+\frac{3}{2}\alpha_0\Big)F_4''+2\eta\Big(1-\alpha_0+\frac{\alpha_0}{\rho}\Big)F_4'+4\Big(1-\alpha_0+\frac{\alpha_0}{\rho}\Big)F_4=4\frac{\alpha_0}{\rho}G_4+2\eta\frac{\alpha_0}{\rho}G_4'+\frac{15\vartheta}{4\alpha^2}\alpha_0M_6'. \qquad [5.17]$$

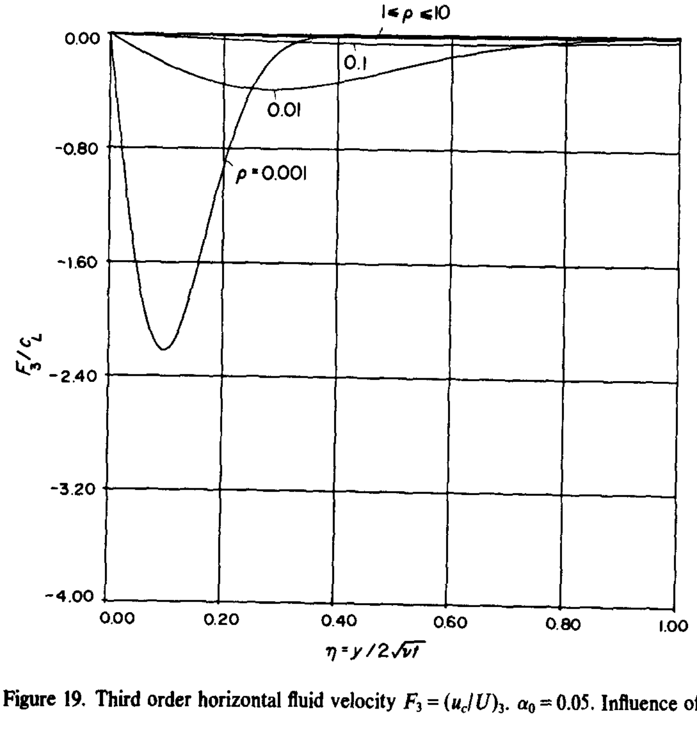

Figure 19. Third order horizontal fluid velocity $F_3=(u_c/U)_3$. $\alpha_0=0.05$. Influence of ρ.

The r.h.s showing the two disturbances causing the fourth order perturbation F_4:

(1) The Stokes drag force due to the slip velocity $G_4 = (\tilde{u}_c - \tilde{u}_d)_4$.
(2) Antisymmetric stresses due to relative rotation M_6 between particles and fluid.

However, the boundary value problem $F_4(0) = F_4(\infty) = 0$ is not unique, since there exist no two linear independent solutions for the l.h.s. of [5.17]. Hence, the numerical solution of [5.17] did not converge. Unless $F_4'(0)$ is known, which may be obtained by an integral formulation of the present problem in a similar way as given by Murray (1967), [5.17] cannot be solved. So we stop the series expansion for $F_\gamma(\eta)$ here.

An analytical solution for the relative rotation between particles and fluid can be obtained from the balance of angular momentum, [3.13]:

$$M_0 = M_1 = M_2 = M_3 = M_4 = M_5 = 0$$

and

$$M_6(\eta) = \frac{3}{80}(F_0' + \eta F_0'') = -\frac{3\beta_{II}}{40\sqrt{\pi}}(1 - 2\beta_{II}^2\eta^2)\exp(-\beta_{II}^2\eta^2). \qquad [5.18]$$

When $F_4(\eta)$ is given, the particle rotation $(\omega_d)_6$ may be obtained from the above

$$(\tilde{\omega}_d)_6 = -\frac{1}{4}F_4'(\eta) + \frac{3\beta_{II}}{40\sqrt{\pi}}(1 - 2\beta_{II}^2\eta^2)\exp(-\beta_{II}^2\eta^2). \qquad [5.19]$$

5.3 *Summary of results for $\tau \gg 1$*

As far as they have been examined by the author, the series expansions [5.1] are given, the terms in brackets being the next nonzero terms not considered here.

$$\begin{aligned}
\tilde{u}_c &= F_0(\eta) + F_3(\eta)\tau^{-3/4} + F_4(\eta)\tau^{-4/4} + \cdots \\
\tilde{u}_c - \tilde{u}_d &= G_4(\eta)\tau^{-4/4} + [G_7(\eta)\tau^{-7/4}] + \cdots \\
\tilde{v}_c &= H_5(\eta)\tau^{-5/4} + [H_7(\eta)\tau^{-7/4}] + \cdots \\
\tilde{v}_c - \tilde{v}_d &= L_5(\eta)\tau^{-5/4} + [L_7(\eta)\tau^{-7/4}] + \cdots \\
\tilde{\omega}_c - \tilde{\omega}_d &= M_6(\eta)\tau^{-6/4} + \cdots \\
\tilde{p}_c &= p_0 + [p_7(\eta)\tau^{-7/4}] + \cdots \\
\alpha &= \alpha_0 + \alpha_3(\eta)\tau^{-3/4} + [\alpha_5(\eta)\tau^{-5/4}] + \cdots .
\end{aligned} \qquad [5.20]$$

6. WALL SHEAR STRESS

The wall shear stress obtained from the constitutive equation for the continuous phase stress tensor, [2.18] and [2.21] in dimensionless form is

$$\tilde{\tau}_w = -\tilde{\tau}_{yx}^c(y = 0) \qquad [6.1]$$

$$\tilde{\tau}_{yx}^c = \left(1 + \frac{5}{2}\alpha + \frac{\alpha}{1-\alpha}\frac{15\vartheta}{4a^2}\right)\frac{\partial\tilde{u}_c}{\partial\tilde{y}} + \frac{\alpha}{1-\alpha}\frac{15\vartheta}{4a^2}2\tilde{\omega}_d \qquad [6.2]$$

with

$$\tilde{\tau}_W = \frac{\sqrt{(\tau_P/\nu)}}{\rho_c U}\tau_W . \qquad [6.3]$$

With the similarity transformation [4.1] one obtains

$$\tilde{\tau}^c_{\eta x} = \left(1+\frac{5}{2}\alpha+\frac{\alpha}{1-\alpha}\frac{15\vartheta}{4a^2}\right)\frac{1}{2\sqrt{\tau}}\frac{\partial \tilde{u}_c}{\partial \eta}+\frac{\alpha}{1-\alpha}\frac{15\vartheta}{4a^2}2\tilde{\omega}_d. \qquad [6.4]$$

The wall shear stress for a pure fluid ($\alpha = 0$) is (see Schlichting 1965)

$$(\tilde{\tau}_W)_{\alpha_0=0} = \frac{1}{\sqrt{\pi}}\tau^{-1/2}. \qquad [6.5]$$

(a) *Small time solution*, $\tau \ll 1$. With the results for $\tau \ll 1$ for $\tilde{u}_c$ and $\tilde{\omega}_d$ as given in section 4 one obtains

$$\frac{\tilde{\tau}_W}{(\tilde{\tau}_W)_{\alpha_0=0}} = \beta_I\left\{\left(1+\frac{5}{2}\alpha_0+\frac{15\vartheta}{4a^2}\alpha_0\right)+\left[\left(1+\frac{5}{2}\alpha_0+\frac{15\vartheta}{4a^2}\alpha_0\right)\frac{Q}{4}-\frac{15\vartheta}{4a^2}\frac{20}{3}\alpha_0\right]\tau+0(\tau^{5/4})\right\}>1. \qquad [6.6]$$

With β_I from [4.9b] and Q from [4.21b] we have

$$\frac{\tilde{\tau}_W}{(\tilde{\tau}_W)_{\alpha_0=0}} = 1+\frac{5}{4}\alpha_0+\frac{15\vartheta}{8a^2}\alpha_0+\frac{1}{4+\rho/2}\alpha_0+\left[\frac{1}{\rho(1+\rho/2)^2}-\frac{50\vartheta}{4a^2}\right]\alpha_0\tau+0(\tau^{5/4})+\cdots \qquad [6.7]$$

where we have expanded β_I and Q in a Taylor series for $\alpha_0 \ll 1$ and terms of $0(\alpha_0^2)$ have been neglected.

Here the influences giving rise to a higher wall shear stress than in a particle free ($\alpha_0 = 0$) fluid are demonstrated. To the first, the increased shear viscosity $\mu_c = \mu(1+(5/2)\alpha_0)$ and to the second the antisymmetric stresses of the continuous phase (term $15\vartheta\alpha_0/4a^2$ in [6.6]) increase the wall shear stress. On the other hand the particle rotation reduces τ_w, as is seen by $-(15\vartheta/4a^2)(20/3)\alpha_0$ in [6.6]. This effect, however, is small, since the particle rotation is of second order ($m_2(\eta) \neq 0$). For the above cited numerical example with $\alpha_0 = 0.05$, $\rho = 0.001$ and for $\tau = t/\tau_P = 0.01$ we have $\tau_W/\tau_{W\alpha_0=0} = 1.1125+49.70\tau = 1.6095$, which shows an increase of 61 per cent of the wall shear stress in comparison to the particle free case.

(b) *Large time solution*, $\tau \gg 1$. The ratio of the wall shear stress to the wall shear stress for a particle free fluid for the large time solution is

$$\frac{\tilde{\tau}_W}{(\tilde{\tau}_W)_{\alpha_0=0}} = \left(1+\frac{5}{2}\alpha_0\right)\left(\beta_{II}-\frac{\sqrt{\pi}}{2}F_3'(0)\tau^{-3/4}+0(\tau^{-4/4})+\cdots\right) \qquad [6.8]$$

with β_{II} from [5.6b] we have

$$\frac{\tilde{\tau}_W}{(\tilde{\tau}_W)_{\alpha_0=0}} = 1+\left(\frac{1}{2\rho}+\frac{5}{4}\right)\alpha_0-\frac{\sqrt{\pi}}{2}\left(1+\frac{5}{2}\alpha_0\right)F_3'(0)\tau^{-3/4}+\cdots. \qquad [6.9]$$

To the third order antisymmetric stresses do not affect the wall shear stress since there is no relative rotation between the phases ($M_0 = M_1 = M_2 = M_3 = M_4 = M_5 = 0$). The wall shear stress is raised in comparison to a particle free fluid by the viscosity $\mu_c = \mu(1+(5/2)\alpha_0)$ and by the density ratio $\rho = \rho_c/\rho_d$. Hence, heavy particles may contribute to an appreciable rise of the wall shear stress (see the term $\alpha_0/2\rho$ in [6.9]).

For $\alpha_0 = 0.05$ and $\rho = 0.001$ this gives for example, $\tau_W/\tau_{W\alpha_0=0} = 27.06+f(\tau^{-3/4})$, showing that the wall shear stress due to the presence of the particles is 27 times greater than for the case without particles.

7. CONCLUSION

The motion of the continuous and the dispersed phase in the flow induced by the impulsive motion of an infinite flat plate has been examined. The series expansion shows that for small times antisymmetric stresses of the continuous phase are important. The antisymmetric stresses are proportional to the relative rotation between the phases, this relative rotation being high for small times due to the rotational intertia of the particles. For small times the zeroth order fluid velocity is raised by the antisymmetric stresses. Also, for small times, antisymmetric stresses contribute to a higher wall shear stress. For large times, when there exists very small relative rotation between the phases, antisymmetric stresses, therefore, are of very small influence.

Hence, antisymmetric stresses of the continuous phase may be important in a suspension flow whenever there are great differences in the relative rotation between the phases. This relative rotation may be caused by inertia forces as in the present problem, by body couples on the particles or by a rigid array of the particles (as in a porous medium).

There are no experiments known to the author concerning impulsive motion of a suspension. The only experiments on a similar flow problem are measurements by Einav & Lee (1973) on particles migration in laminar boundary layer flow. Due to the restriction to heavy particles for the small time solution experiments to test the small time solution should be done in vertical flow along a vertical plate. The inclusion of gravity in the above calculations does not introduce mathematical difficulties.

Another flow situation where antisymmetric stresses may be important, is the case when the flat plate is oscillating (Liu 1966), a flow situation which occurs, for example in viscometric flows, where viscosity components of suspensions are measured with an oscillating cone and plate viscometer (Chien *et al.* 1975).

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